Geometric Group Theory (L16)

Non-Examinable (Part III Level)

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Groups are algebraic objects which are well-suited to capturing notions of symmetry. As well as their intrinsic algebraic structure, groups have the ability to act on other mathematical objects such as sets or spaces. This action is often useful for learning more about either the object that the group is acting on, or the group itself. The Orbit-Stabiliser Theorem, which students will have already met, is a basic example of this phenomenon.

When a group acts on a metric space in a sufficiently nice way, one can often use the geometric properties of the space to deduce algebraic or analytic information about the group. In this way, one can build up a dictionary between algebra and geometry, which features beautiful and sometimes surprising connections between these two subjects. These connections can often be exploited to solve deep problems in other fields such as topology or analysis.

One metric space on which a group acts in a particularly pleasing way can be created using data from the group itself. Namely, by fixing a generating set of the group, one can construct a graph with vertex set equal to the set of elements of the group, with edges defined using multiplication by elements of the generating set. This graph, called a Cayley graph of the group, is not only a neat visualisation of the group, but is also an invaluable tool in modern group theory, since the geometric properties of this graph are profoundly connected to the group-theoretic properties of the group. Geometric group theory is the study of groups and spaces via these connections.

In this course, we will concentrate on some of the following aspects of this rich theory (time permitting):

- Cayley graphs; quasi-isometries; the Švarc–Milnor Lemma;
- a Smörgåsbord of geometric properties and invariants of groups, such as growth, ends, hyperbolicity, and connections to algorithmic group theory;
- analytic properties of groups, such as amenability, and connections to actions on Banach spaces.

Pre-requisites

A good knowledge of basic group theory is essential. Part II Algebraic Topology (or equivalent) is required, and some intuition in graph theory and geometry would be helpful. Some functional analysis (such as the Part II Linear Analysis course or the beginning of the Part III Functional Analysis course) will be useful for the last part of the course.

Literature

1. P. de la Harpe, *Topics in Geometric Group Theory*, Chicago Lectures in Mathematics, 2000.
2. C. Druţu and M. Kapovich, *Geometric Group Theory*, Colloquium Publications 63, 2018. Also available at

   https://www.math.ucdavis.edu/~kapovich/EPR/ggt.pdf

3. P. W. Nowak and G. Yu, *Large Scale Geometry*, EMS Textbooks in Mathematics, 2012.

4. M. Clay and D. Margalit (Editors), *Office Hours with a Geometric Group Theorist*, Princeton University Press, 2017.

5. M. R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Grundlehren der Mathematischen Wissenschaften 319, 1999.