Faculty of Mathematics
Part III Essays: 2018-19

Titles 1 – 56
Department of Pure Mathematics
& Mathematical Statistics

Titles 57 – 95
Department of Applied Mathematics
& Theoretical Physics

Titles 96 – 117
Additional Essays
Contents

Introductory Notes .......................................................... 8

1. The Congruence Subgroup Problem ........................................... 16

2. Non-Arithmetic Lattices ........................................................ 16

3. Canonical Kähler Metrics on Projective Varieties ......................... 17

4. Modular Curves and the Class Number One Problem ..................... 18

5. Classical Invariant Theory and Moduli of Genus 2 Curves ............... 18

6. Fraenkel-Mostowski Models for Set Theory ................................ 19

7. The Probability that a Random Matrix is Singular ....................... 20

8. Probabilistically Checkable Proofs ......................................... 20

9. Riemann-Hilbert Correspondence for Differential Equations with Regular and Irregular Singularities ........................................ 21

10. Quantum Groups, KZ Equations, $Gal(\bar{Q}/Q)$, Periods ............... 22

11. The Foundations of Logarithmic Geometry .................................. 23

12. Algebraic Stacks ............................................................ 24

13. Tropical Geometry .......................................................... 25

14. Classifying Toposes .......................................................... 25

15. Locally Presentable and Accessible Categories ............................. 26

16. Synthetic Differential Geometry ............................................ 27

17. Lagrangian Tori in $\mathbb{R}^6$ ............................................. 28

18. Complements of Hyperplane Arrangements ................................ 28

19. Yau’s Solution of the Calabi Conjecture ................................... 29

20. Dirac Operators ............................................................. 30

21. (No) Wandering Domains .................................................... 30
22. Effective Diophantine Approximation and Unlikely Intersections .......... 31
23. Independence Results for Basic Axioms of Set Theory ..................... 32
24. The Generalised Real Numbers ............................................. 33
25. Long Blackwell Games .................................................... 33
26. Group Cohomology from the Topological Viewpoint ........................ 34
27. Surface Bundles .......................................................... 35
28. $E_k$-algebras ............................................................ 36
29. The Heegaard Floer Contact Invariant ..................................... 36
30. Annular Khovanov Homology ................................................. 37
31. $p$-adic Modular Forms ...................................................... 38
32. Symplectic Structures on Euclidean Space .................................. 38
33. Packing Symplectic Tori ...................................................... 39
34. Dynamics on K3 Surfaces ..................................................... 40
35. Expansion and Robust Expansion ........................................... 40
36. Sidorenko’s Conjecture ....................................................... 41
37. Cohomology of Number Fields .............................................. 41
38. Arithmetic Statistics of Elliptic Curves .................................... 42
39. Komlós’s Conjecture in Discrepancy Theory ................................ 43
40. Equidistribution of Roots of Polynomials ................................... 43
41. Croot-Sisask Almost-Periodicity and Applications .......................... 44
42. Concentration and Functional Inequalities and their Relation to Markov Processes ......................................................... 45
43. Piecewise Deterministic Markov Processes .................................. 45
44. Random Matrix Eigenvalue Statistics ....................................... 46
45. Loop Erased Random Walk and SLE$_2$ ........................................ 47
46. Random Walk on Super Critical Percolation Clusters ......................... 47
47. Random Walks on Height Functions .................................................. 48
48. Brownian Motion on a Riemannian Manifold ..................................... 49
49. Optimal Allocation in Sequential Multi-armed Clinical Trials with a Binary Response ................................................................. 50
50. The EM and $k$-means Algorithms ................................................... 50
51. Applications of Random Matrix Theory in Statistics ............................ 51
52. Estimation of Heterogeneous Treatment Effects ................................... 52
53. Recent Developments in False Discovery Rate Control ......................... 53
54. Statistical Inference Using Machine Learning Methods ....................... 54
55. Model-Free No-Arbitrage Bounds .................................................. 55
56. Polynomial Preserving Processes .................................................. 56
57. Precision Higgs Mass Predictions in Minimal Supersymmetry ................ 56
58. Edge-Turbulence Interaction and the Generation of Sound .................... 57
59. Extrema in Gaussian Random Fields as a Proxy for Galaxy Clustering ... 58
60. Dualities and the Equivalence of Physical Theories ............................. 59
61. Symmetry and Symplectic Reduction ............................................. 61
62. Viscoelastic Instabilities in Soft Matter ........................................ 62
63. Mixing Efficiency in Stratified Fluids ............................................. 63
64. Instability and Perturbation Growth in Stratified Shear Flows ................ 64
65. Recovering Quantum Information ................................................. 65
66. Polynomial Optimization on the Sphere ........................................ 66
67. ‘Moisture Modes’ and the Tropical Atmosphere .................................. 66
<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>91.</td>
<td>Lattice QCD and Hadron Spectroscopy</td>
<td>85</td>
</tr>
<tr>
<td>92.</td>
<td>Chiral Fermions on the Lattice</td>
<td>86</td>
</tr>
<tr>
<td>93.</td>
<td>Strichartz Estimates and Nonlinear Schrödinger Equations</td>
<td>87</td>
</tr>
<tr>
<td>94.</td>
<td>Linear Fields on Black Hole Backgrounds</td>
<td>87</td>
</tr>
<tr>
<td>95.</td>
<td>Stationary Spacetimes and Finsler Geometry</td>
<td>88</td>
</tr>
<tr>
<td>96.</td>
<td>Markov Chain Monte Carlo for Tall Data</td>
<td>89</td>
</tr>
<tr>
<td>97.</td>
<td>Chaining and Metric Entropy</td>
<td>89</td>
</tr>
<tr>
<td>98.</td>
<td>Instantons</td>
<td>90</td>
</tr>
<tr>
<td>99.</td>
<td>Statistical Inference for Discretely Observed Compound Poisson Processses and Related Jump Processes</td>
<td>91</td>
</tr>
<tr>
<td>100.</td>
<td>On the structure of Chevalley groups over local fields</td>
<td>92</td>
</tr>
<tr>
<td>101.</td>
<td>Kodaira’s problem</td>
<td>93</td>
</tr>
<tr>
<td>102.</td>
<td>Semipositivity properties of the tangent bundle</td>
<td>94</td>
</tr>
<tr>
<td>103.</td>
<td>Introduction to the Minimal Model Programme</td>
<td>95</td>
</tr>
<tr>
<td>104.</td>
<td>Pursuit on Graphs</td>
<td>96</td>
</tr>
<tr>
<td>105.</td>
<td>Hamiltonian Cycles and Spheres in Hypergraphs</td>
<td>96</td>
</tr>
<tr>
<td>106.</td>
<td>Gravity currents passing over cavities</td>
<td>97</td>
</tr>
<tr>
<td>107.</td>
<td>Clifford algebras and their connection to elementary particle physics</td>
<td>98</td>
</tr>
<tr>
<td>108.</td>
<td>Modelling Quantum Dynamics Using Random Unitary Circuits</td>
<td>98</td>
</tr>
<tr>
<td>109.</td>
<td>Sheaves on Locales and Internal Locales</td>
<td>99</td>
</tr>
<tr>
<td>110.</td>
<td>Conformal Field Theories</td>
<td>100</td>
</tr>
<tr>
<td>111.</td>
<td>Stochastic Graphical Games</td>
<td>100</td>
</tr>
<tr>
<td>112.</td>
<td>Stretch factors of pseudo-Anosovs</td>
<td>101</td>
</tr>
<tr>
<td>113.</td>
<td>The Search for CMB B-mode Polarization from Inflationary Gravitational Waves</td>
<td>102</td>
</tr>
</tbody>
</table>
114. Puffs of bubbles in a stratified environment .........................103
115. Strongly interacting gravity currents ................................. 104
116. Ultra slow-roll inflation .............................................. 104
117. Bounded gaps between primes ....................................... 105
Introductory Notes

General advice. Before attempting any particular essay, candidates are advised to meet the setter in person. Normally candidates may consult the setter up to three times before the essay is submitted. The first meeting may take the form of a group meeting at which the setter describes the essay topic and answers general questions.

Choice of topic. The titles of essays appearing in this list have already been announced in the Reporter. If you wish to write an essay on a topic not covered in the list you should approach your Part III Adviser or any other member of staff to discuss a new title. You should then ask your Director of Studies to write to the Secretary of the Faculty Board, c/o the Undergraduate Office at the CMS (Room B1.28) no later than 1 February. The new essay title will require the approval of the Examiners. It is important that the essay should not substantially overlap with any course being given in Part III. Additional Essays will be announced in the Reporter no later than 1 March and are open to all candidates. Even if you request an essay you do not have to do it. Essay titles cannot be approved informally: the only allowed essay titles are those which appear in the final version of this document (on the Faculty web site).

Originality. The object of a typical essay is to give an exposition of a piece of mathematics which is scattered over several books or papers. Originality is not usually required, but often candidates will find novel approaches. All sources and references used should be carefully listed in a bibliography.

Length of essay. There is no prescribed length for the essay in the University Ordinances, but the general opinion seems to be that 5,000-8,000 words is a normal length. If you are in any doubt as to the length of your essay please consult your adviser or essay setter.

Presentation. Your essay should be legible and may be either hand written or produced on a word processor. Candidates are reminded that mathematical content is more important than style. Usually it is advisable for candidates to write an introduction outlining the contents of the essay. In some cases a conclusion might also be required. It is very important that you ensure that the pages of your essay are fastened together in an appropriate way, by stapling or binding them, for example.

Credit. The essay is the equivalent of one three-hour exam paper and marks are credited accordingly.

Final decision on whether to submit an essay. You are not asked to state which papers you have chosen for examination and which essay topic, if any, you have chosen until the beginning of the third term (Easter) when you will be sent the appropriate form to fill in and hand to your Director of Studies. Your Director of Studies should counter-sign the form and send it to the Chairman of Examiners (c/o the Undergraduate Office, Centre for Mathematical Sciences) so as to arrive not later than 12 noon of the second Thursday in Easter Full Term, which this year is Thursday 2 May 2019. Note that this deadline will be strictly adhered to.
Date of submission. You should submit your essay to the Chairman of Examiners (c/o Undergraduate Office, CMS). Your essay should be sent with the completed essay submission form found on page 11 of this document. The form should be completed and signed by you. Please do not bind or staple the essay submission form to your essay, but instead attach it loosely, e.g. with a paperclip.

Then you should take your essay and the signed essay submission form to the Undergraduate Office (B1.28) at the Centre for Mathematical Sciences so as to arrive not later than 12 noon of the second Thursday in Easter Full Term, which this year is Thursday 2 May 2019. Note that this deadline will be strictly adhered to. If an extension is likely to be needed due to exceptional and unexpected developments, a letter of application and explanation demonstrating the nature of such developments is required from the candidate’s Director of Studies. This application should be sent to the Director of Taught Postgraduate Education by the submission date as detailed above. It is expected that such an extension would be (at most) to the following Monday at 12 noon. A student who is dissatisfied with the decision of the Director of Taught Postgraduate Education can request within 7 days of the decision, or by the submission date (extended or otherwise), whichever is earlier, that the Chair of the Faculty review the decision. The provision of any such extension will be reported to the examiners for Part III.

Title page. The title page of your essay should bear ONLY the essay title. Please DO NOT include your name or any other personal details on the title page or anywhere else on your essay.

Signed declaration. The essay submission form requires you to sign the following declaration. It is important that you read and understand this before starting your essay.

I declare that this essay is work done as part of the Part III Examination. I have read and understood the Statement on Plagiarism for Part III and Graduate Courses issued by the Faculty of Mathematics, and have abided by it. This essay is the result of my own work, and except where explicitly stated otherwise, only includes material undertaken since the publication of the list of essay titles, and includes nothing which was performed in collaboration. No part of this essay has been submitted, or is concurrently being submitted, for any degree, diploma or similar qualification at any university or similar institution.

Important note. The Statement on Plagiarism for Part III and Graduate Courses issued by the Faculty of Mathematics is reproduced starting on page 12 of this document. If you are in any doubt as to whether you will be able to sign the above declaration you should consult the member of staff involved in the essay. If they are unsure about your situation they should consult the Chairman of the Examiners as soon as possible. The examiners have the power to examine candidates viva voce (i.e. to give an oral examination) on their essays, although this procedure is not often used. However, you should be aware that the University takes a very serious view of any use of unfair means (plagiarism, cheating) in University examinations. The powers of the University Court of Discipline in such cases extend to depriving a student of membership of the University. Fortunately, incidents of this kind are very rare.

Return of essays. It is not possible to return essays. You are therefore advised to make your own copy before handing in your essay.
**Further advice.** It is important to control carefully the amount of time spent writing your essay since it should not interfere with your work on other courses. You might find it helpful to construct an essay-writing timetable with plenty of allowance for slippage and then try your hardest to keep to it.

**Research.** If you are interested in going on to do research you should, if possible, be available for consultation in the next few days after the results are published. If this is not convenient, or if you have any specific queries about PhD admissions, please contact the following addresses:

**Applied Mathematics & Theoretical Physics**  
research@damtp.cam.ac.uk  
DAMTP PhD Admissions,  
Mathematics Graduate Office,  
Centre for Mathematical Sciences,  
Wilberforce Road,  
Cambridge CB3 0WA,  
United Kingdom.

**Pure Mathematics & Mathematical Statistics**  
research@dpmms.cam.ac.uk  
DPMMS PhD Admissions,  
Mathematics Graduate Office,  
Centre for Mathematical Sciences,  
Wilberforce Road,  
Cambridge CB3 0WB,  
United Kingdom.
To the Chairman of Examiners for Part III Mathematics.

I declare that this essay is work done as part of the Part III Examination. I have read and understood the Statement on Plagiarism for Part III and Graduate Courses issued by the Faculty of Mathematics, and have abided by it. This essay is the result of my own work, and except where explicitly stated otherwise, only includes material undertaken since the publication of the list of essay titles, and includes nothing which was performed in collaboration. No part of this essay has been submitted, or is concurrently being submitted, for any degree, diploma or similar qualification at any university or similar institution.

Signed: ............................ Date: ............................

Title of Essay: ............................

Essay Number: ............................

Name: ............................ College: ............................

Assessor comments:
Your home address is needed to return any essay comments we receive, which will be sent out in June/July 2019. Comments are not mandatory and your assessor may not provide them. Please supply a self-addressed envelope or provide your home address below.

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Spare envelopes are also available in the Undergraduate Office.
Appendix: Faculty of Mathematics Guidelines on Plagiarism

For the latest version of these guidelines please see http://www.maths.cam.ac.uk/facultyboard/plagiarism/

University Resources

The University publishes information on *Good academic practice and plagiarism*, including

- a *University-wide statement on plagiarism*;
- Information for students, covering
  - *Your responsibilities*
  - *Why does plagiarism matter?*
  - *Collusion*
- information about *Referencing* and *Study skills*;
- information on *Resources and support*;
- the *University’s statement on proofreading*;
- *FAQs*.

There are references to the University statement

- in the *Part IB and Part II Computational Project Manuals*,
- in the *Part III Essay booklet*, and
- in the *M.Phil. Computational Biology Course Guide*.

*Please read the University statement carefully; it is your responsibility to read and abide by this statement.*

The Faculty Guidelines

The guidelines below are provided by the Faculty to help students interpret what the University Statement means for Mathematics. However neither the University Statement nor the Faculty Guidelines supersede the University’s Regulations as set out in the *Statutes and Ordinances*. If you are unsure as to the interpretation of the University Statement, or the Faculty Guidelines, or the *Statutes and Ordinances*, you should ask your Director of Studies or Course Director (as appropriate).

What is plagiarism?

Plagiarism can be defined as the *unacknowledged use of the work of others as if this were your own original work*. In the context of any University examination, this amounts to *passing off the work of others as your own to gain unfair advantage*.

Such use of unfair means will not be tolerated by the University or the Faculty. If detected, the penalty may be severe and may lead to failure to obtain your degree. This is in the interests of the vast majority of students who work hard for their degree through their own efforts, and it is essential in safeguarding the integrity of the degrees awarded by the University.
Checking for plagiarism

Faculty Examiners will routinely look out for any indication of plagiarised work. They reserve the right to make use of specialised detection software if appropriate (the University subscribes to Turnitin Plagiarism Detection Software). See also the Board of Examinations’ statement on How the University detects and disciplines plagiarism.

The scope of plagiarism

Plagiarism may be due to

- **copying** (this is using another person’s language and/or ideas as if they are your own);
- **collusion** (this is collaboration either where it is forbidden, or where the extent of the collaboration exceeds that which has been expressly allowed).

How to avoid plagiarism

Your course work, essays and projects (for Parts IB, II and III, the M.Phil. etc.), are marked on the assumption that it is your own work: i.e. on the assumption that the words, diagrams, computer programs, ideas and arguments are your own. Plagiarism can occur if, without suitable acknowledgement and referencing, you take any of the above (i.e. words, diagrams, computer programs, ideas and arguments) from books or journals, obtain them from unpublished sources such as lecture notes and handouts, or download them from the web.

Plagiarism also occurs if you submit work that has been undertaken in whole or part by someone else on your behalf (such as employing a ‘ghost writing service’). Furthermore, you should not deliberately reproduce someone else’s work in a written examination. These would all be regarded as plagiarism by the Faculty and by the University.

In addition you should not submit any work that is substantially the same as work you have submitted, or are concurrently submitting, for any degree, diploma or similar qualification at any university or similar institution.

However, it is often the case that parts of your essays, projects and course-work will be based on what you have read and learned from other sources, and it is important that in your essay or project or course-work you show exactly where, and how, your work is indebted to these other sources. The golden rule is that the Examiners must be in no doubt as to which parts of your work are your own original work and which are the rightful property of someone else.

A good guideline to avoid plagiarism is not to repeat or reproduce other people’s words, diagrams or computer programs. If you need to describe other people’s ideas or arguments try to paraphrase them in your own words (and remember to include a reference). Only when it is absolutely necessary should you include direct quotes, and then these should be kept to a minimum. You should also remember that in an essay or project or course-work, it is not sufficient merely to repeat or paraphrase someone else’s view; you are expected at least to evaluate, critique and/or synthesise their position.

In slightly more detail, the following guidelines may be helpful in avoiding plagiarism.

**Quoting.** A quotation directly from a book or journal article is acceptable in certain circumstances, provided that it is referenced properly:
• short quotations should be in inverted commas, and a reference given to the source;
• longer pieces of quoted text should be in inverted commas and indented, and a reference given to the source.

Whatever system is followed, you should additionally list all the sources in the bibliography or reference section at the end of the piece of work, giving the full details of the sources, in a format that would enable another person to look them up easily. There are many different styles for bibliographies. Use one that is widely used in the relevant area (look at papers and books to see what referencing style is used).

**Paraphrasing.** Paraphrasing means putting someone else’s work into your own words. Paraphrasing is acceptable, provided that it is acknowledged. A rule of thumb for acceptable paraphrasing is that an acknowledgement should be made at least once in every paragraph. There are many ways in which such acknowledgements can be made (e.g. “Smith (2001) goes on to argue that ...” or “Smith (2001) provides further proof that ...”). As with quotation, the full details of the source should be given in the bibliography or reference list.

**General indebtedness.** When presenting the ideas, arguments and work of others, you must give an indication of the source of the material. You should err on the side of caution, especially if drawing ideas from one source. If the ordering of evidence and argument, or the organisation of material reflects a particular source, then this should be clearly stated (and the source referenced).

**Use of web sources.** You should use web sources as if you were using a book or journal article. The above rules for quoting (including ‘cutting and pasting’), paraphrasing and general indebtedness apply. Web sources must be referenced and included in the bibliography.

**Collaboration.** Unless it is expressly allowed, collaboration is collusion and counts as plagiarism. Moreover, as well as not copying the work of others you should not allow another person to copy your work.

**Links to University Information**

• Information on *Plagiarism and good academic practice*, including
  • *Students’ responsibilities*.
  • *Information for staff*.
Table 1: A Timetable of Relevant Events and Deadlines

<table>
<thead>
<tr>
<th>Date</th>
<th>Event Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Friday 1 February</td>
<td>Deadline for Candidates to request additional essays.</td>
</tr>
<tr>
<td>Thursday 2 May, noon</td>
<td>Deadline for Candidates to return form stating choice of papers and essays.</td>
</tr>
<tr>
<td>Thursday 2 May, noon</td>
<td>Deadline for Candidates to submit essays.</td>
</tr>
<tr>
<td>Thursday 30 May</td>
<td>Part III Examinations begin.</td>
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Comments. If you feel that these notes could be made more helpful please write to The Chairman of Examiners, c/o the Undergraduate Office, CMS.

Further information. Professor T.W. Körner (DPMMS) wrote an essay on Part III essays which may be useful (though it is slanted towards the pure side). It is available via his home page

1. The Congruence Subgroup Problem .................................................. Professor E. F. J. Breuillard

This is a famous group problem in group theory aiming at understanding finite index subgroups of\textit{arithmetic groups}. Mennicke and Bass-Milnor-Serre proved that every finite index subgroup of $SL_n(\mathbb{Z})$, $n \geq 3$ contains a congruence subgroup, namely the kernel of the reduction modulo $N$ homomorphism $SL_n(\mathbb{Z}) \to SL_n(\mathbb{Z}/N\mathbb{Z})$ for some integer $N$. This question can be formulated more generally for an arbitrary arithmetic group and a conjecture of Serre describes what is expected.

The essay would focus on the proof for $SL_n(\mathbb{Z})$ first and then venture to other territories at the student’s discretion.

Relevant Courses

\textit{Essential:} None

\textit{Useful:} Part II: Representation theory, Part III: Algebra, Graduate: Discrete subgroups of Lie groups.

References


2. Non-Arithmetic Lattices ................................................................. Professor E. F. J. Breuillard

A lattice in a Lie group is a discrete group of finite co-volume. A celebrated theorem of Margulis asserts that every lattice in a $SL_n(\mathbb{R})$, $n \geq 3$, is arithmetic. More generally every irreducible lattice in a semisimple Lie group of rank at least 2 is arithmetic. This was extended by Corlette and Gromov-Schoen to all rank one Lie groups, except for the families $SO(n, 1)$ and $SU(n, 1)$, which correspond respectively to groups of isometries of real and complex hyperbolic spaces.

Non-arithmetic lattices in $SO(n, 1)$ have been constructed as certain reflection groups for certain $n$ and a general construction has been given by Gromov and Piatetski-Shapiro. In $SU(n, 1)$, $n \geq 2$ only finitely many examples are known of non-arithmetic lattices and none are known for $n \geq 4$.

The essay would aim at giving an exposition of the Gromov and Piatetski-Shapiro examples in the first place, and then move to the Mostow examples of non-arithmetic lattices in $SU(2, 1)$ and beyond if time permits.

Relevant Courses

\textit{Essential:} Part II: Differential Geometry

\textit{Useful:} Part III: Algebraic topology, Graduate: Discrete subgroups of Lie groups.
3. Canonical Kähler Metrics on Projective Varieties

The natural candidate for a “best” metric on a smooth projective variety is a Kähler metric with constant scalar curvature. The existence problem for such metrics is subtle, with the guiding principle being Donaldson’s conjecture that the existence of such metrics should be equivalent to the algebro-geometric notion of K-stability. The goal of this essay is to discuss Donaldson’s proof of one direction of this conjecture: the existence of a constant scalar curvature Kähler metric implies K-semistability [1].

The essay should begin with the definition of constant scalar curvature Kähler metrics, as given in [2]. The essay should next discuss the Hilbert polynomial and weight polynomials of projective varieties, including a proof that these are indeed polynomials (proven for example in [3, Theorem 9.1 and Proposition 3.12]). Next the essay should define K-semistability following [1]. The bulk of the essay should consist of proving [2, Theorem 1]. Some parts of Donaldson’s proof are only sketched, and the essay should contain a clear account of the claim that the function \( f(t) \) used in [2, p464] is increasing; one exposition of this is [2, Lemma 7.19]. The proof of [2, Proposition 3] given by Donaldson is technically challenging and an alternative proof can be found in [4, Theorem 27]. The expansion of the density of states function \( \rho_k \) used by Donaldson should be stated clearly, but the proof of this expansion should be left as a black box.

An ambitious author may wish to give an example of a K-unstable variety, for example through the construction given in [2, Section 6.5].

Relevant Courses

Essential: Algebraic Geometry, Complex Manifolds

References

4. Modular Curves and the Class Number One Problem

Dr T. A. Fisher

The class number one problem of Gauss asks for the complete list of imaginary quadratic fields with class number one. It has long been known that there are at least nine such fields. The first proofs that this list is complete were given by Baker (using linear forms in logarithms) and Stark (using modular functions) in 1966. This essay should describe the latter approach, which is related to earlier work of Heegner. The starting point should be the treatment of Serre [6, Sections A.5 and A.6], who reduces the problem to that of determining the integral points on a certain modular curve \( X_0^+(N) \). Nowadays the proof can be completed in several different ways, that is, by considering different values of \( N \); see [1], [2], [4] and [5].

Relevant Courses

**Essential:** None

**Useful:** Elliptic Curves, Algebraic Number Theory

References


5. Classical Invariant Theory and Moduli of Genus 2 Curves

Dr T. A. Fisher

This essay should begin by reviewing the classical (i.e. 19th century) invariant theory of binary forms of degree \( n \) with particular reference to the cases \( n = 4 \) and \( n = 6 \). The case \( n = 4 \) is related to elliptic curves (see [7]), which are classified up to isomorphism (over an algebraically closed field) by their \( j \)-invariant. The case \( n = 6 \) leads to the definition of the Igusa (or Igusa-Clebsch) invariants that likewise classify genus 2 curves. The main aim of the essay should be to describe the algorithm of Mestre [4] for recovering the equation for a genus 2 curve from its Igusa invariants. If time and space permit, then the connection to Siegel modular forms, or applications such as those in [6], could be discussed.
Relevant Courses

*Essential:* None

*Useful:* Elliptic Curves, Algebraic Geometry

References


6. Fraenkel-Mostowski Models for Set Theory .................................

Dr T. Forster

Fraenkel-Mostowski models were developed by Fraenkel, Mostowski and Specker, originally as a means of demonstrating the independence of the Axiom of Choice from the other axioms of set theory. Later refinements were able to tease apart various choice-like principles: for example one can show that none of the implications in the chain: AC $\rightarrow$ “Every partial order can be refined to a total ordering” $\rightarrow$ “every set can be totally ordered” $\rightarrow$ “Every set of finite sets has a choice function” can be reversed.

In each case the sets of an FM model are sets that are in a suitable sense invariant under the action of a judiciously chosen group of permutations of atoms; so there is a bit of group theory involved, and of course a bit of Set Theory. These same ideas of invariance are of course in play in the “forcing” proofs of independence exhibited later by Cohen (and which are touched on in Part III “Topics in Set Theory”) but the focus there is on the forcing and the purpose of this essay is rather to study the invariance.

There is a variety of applications/aspects which the student can choose between. There are the independence proofs of course; there are recent developments in Set Theory using FM techniques (without forcing); there is a nice semantics for capture-avoiding substitution (google “Fresh ML”), and the corpus of FM work cries out for an abstract general treatment—and such a project might appeal to students who are inclined towards Category theory.

Relevant Courses

*Essential:* Part II Logic and Set Theory or equivalent.

*Useful:* Undergraduate Topology; Part III Category theory; Part III Topics in Set Theory.
Let $A$ be a random $n \times n$ matrix with $\pm 1$ entries. What is the probability that $A$ is singular? One way that it can be singular is if two of its rows are equal or add to zero, and the probability that this happens is approximately $n(n-1)2^{-n}$. It is conjectured that this possibility is the main way that singularity of $A$ occurs: that is, there is a conjectured upper bound of $(1+o(1))n^22^{-n}$.

This seems to be a hard conjecture. Even proving that the probability tends to zero with $n$ was a significant achievement, due to Komlós, and obtaining an exponential upper bound was a further breakthrough, due to Kahn, Komlós and Szemerédi. Their bound was $O(0.999^n)$.

The purpose of this essay is to present a proof of an upper bound of $(3/4 + o(1))^n$, which was proved by Tao and Vu. Rather surprisingly, their proof involved tools from additive combinatorics, including Freiman’s theorem: a key aim of the essay should be to make it clear to the reader why Freiman’s theorem has any relevance to the problem.

**References**


8. **Probabilistically Checkable Proofs**

**Professor W. T. Gowers**

Unless $P=NP$, which most people believe is not the case, there is no polynomial-time algorithm for determining whether a graph contains a clique of a given size. However, there is a polynomial-time algorithm for checking whether a given set of vertices spans a clique: one just checks that all the pairs of vertices in the set are joined by edges.

Suppose that you did not insist on 100% certainty that a graph contained a clique, but merely on 99.999% certainty. A major, and very surprising, result in theoretical computer science is
the PCP theorem, which roughly speaking states that there is a way of converting a graph into a string of bits in polynomial time such that if you look at a constant number of those bits at random, then you can do a test that will always fail if the graph does not contain a clique and will pass with probability at least 1/2 if it does contain a clique. By repeating the test, this 1/2 can be boosted as close as you like to 1. The first proof of this theorem was very long and complicated, but more recently, in another important development, Irit Dinur found a much more accessible proof. The main purpose of this essay is to present that proof. A secondary purpose is to explain the relationship between the PCP theorem and results that say that if \( P \neq NP \), then certain computational problems are not just hard to solve exactly, but hard even to solve approximately. For example, there is no polynomial-time algorithm that outputs “yes” if a graph contains a clique of size \( m_1 \) and “no” if it doesn’t contain a clique of size \( m_2 \), even when \( m_2 \) is much smaller than \( m_1 \).

Relevant Courses

None, but the proof is combinatorial in flavour.

References


9. Riemann-Hilbert Correspondence for Differential Equations with Regular and Irregular Singularities .................................

Professor I. Grojnowski

The aim of this essay is to understand the basics of the Riemann Hilbert correspondence, which is about solutions of linear differential equations in many variables.

It will be useful if you care about non-commutative geometry and mirror symmetry, number theory, or representation theory and the Langlands programme. It might also be useful if you care about differential equations.

An extremely ambitious essay would include an account of the theorem of Mochizuki and Kedlaya. An account of the paper of Katz would still be a very good essay.

Begin with the basics of vector bundles with flat connection on a projective algebraic variety that are allowed regular singularities along a normal crossing divisor.

The one dimensional version of this is the 19th century theory of Fuchsian differential equations; the higher dimensional version is a theorem of Deligne’s. You can read about this in the articles of Haefliger and Malgrange in Borel’s book.

Then learn the theory of irregular singularities, and Stokes structures, again in the one dimensional case. Kac’s article is a good source, as is Malgrange’s books and papers. (There are also classic textbooks on differential equations...)

To understand the statement and proof of the higher dimensional versions, you will need some notions from birational algebraic geometry. You may also wish to learn about D-modules.

If you are interested in this essay, we should discuss the best way in to the subject for you.

The eventual goal is to read and understand the papers of Sabbah, Mochizuki or Kedlaya, and the consequences of this.
References

Many textbook expositions of D-modules now exist. The two best are by the originators of the subject—Kashiwara and Bernstein (the latter are printed notes, available on the web somewhere).

Borel’s book, Algebraic D-modules, is not a good place to read the material, except for the articles by Haefliger and Malgrange, which are excellent. Katz’s article is


The following articles will be completely incomprehensible to you to start — understanding them is the end goal. Don’t browse them and be put off!


10. Quantum Groups, KZ Equations, Gal(\overline{Q}/Q), Periods ........................

Professor I. Grojnowski

In the mid 1980s Drinfeld and Jimbo defined quantum groups, algebras $U_h$ over $\mathbb{C}[[h]]$ which are deformations of the enveloping algebra $U_0$ of a semisimple Lie algebra.

More precisely, $U_h$ is a Hopf algebra, free over $\mathbb{C}[[h]]$, such that $U_h/hU_h$ is the enveloping algebra. Now, it is easy to see (and you will, in this essay!), that enveloping algebras of semisimple Lie algebras cannot deform — $U_h$ is isomorphic to $U_0 \otimes \mathbb{C}[[h]]$, so one can think of this as saying what is actually changing is how you make the tensor product of two $U_h$-modules a $U_h$-module.

One way of doing this, invented by Drinfeld, is to change the associativity constraint, that is the isomorphism $V_1 \otimes (V_2 \otimes V_3) \simeq (V_1 \otimes V_2) \otimes V_3$ between the tensor product of three modules. And one way of doing this is to use the monodromy of the Knihzik-Zamalodchikov equation, an explicit flat connection on vector bundles over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

The first main goal of this essay is to understand this precisely — that is, to understand Drinfeld’s theorem computing the deformations of $U_0$ as a bialgebra, the construction of the KZ associator, and the Drinfeld-Kohno theorem.

This can all be found in the original papers, or in the lovely textbook by Etingof-Schiffmann.

You may then continue in various ways. You can learn more about the KZ-equation, the connection with unipotent local systems, special values of multi-zeta functions, and the Galois group of $\overline{Q}/Q$.

Or you may prefer to learn about more recent approaches to deformation theory — Kontsevich’s theorem, and its various proofs, and then the topologist’s hierarchy of homotopy commutative algebras, the $E_n$-algebras.
Or you may prefer to understand properties of quantum groups which are not explained by
deformation theory — the quantum groups are defined over \( \mathbb{Z}[q, q^{-1}] \), not just the formal completion of this at \( q = e^{ih} = 1 \).

References


11. The Foundations of Logarithmic Geometry

Professor M. Gross

The theory of logarithmic schemes was developed in the 1980s by Illusie-Fontaine and Kazuya Kato. As described by Kato, a logarithmic structure on a scheme is a “magic powder” which makes relatively nice singular schemes look smooth. A typical example is a normal crossings divisor, which formally looks smooth if viewed as a log scheme.

While the original motivation for introducing log schemes was for its arithmetic applications, more recently log schemes have found powerful applications in mirror symmetry. This essay should cover the fundamentals of log geometry, and then explore applications of interest to the essay writer.

The original papers [1], [2] are dense but readable. Chapter 3 of [3] contains a more relaxed introduction to parts of the theory needed for mirror symmetry. [4] is a partial encyclopedic manuscript on the subject. These sources are more than enough to get started. Further avenues can be explored once the basics are mastered, following up in the mirror symmetry direction via [3] and references therein, or in the arithmetic direction.

Relevant Courses

*Essential:* Part III Algebraic Geometry

References


12. Algebraic Stacks .......................... Professor M. Gross

Algebraic stacks are a vast generalization of the notion of scheme, developed partly to describe various moduli spaces. For example, $\mathcal{M}_g$, the moduli space of algebraic curves of genus $g$, cannot be described as a scheme, but is what is known as a Deligne-Mumford stack. Morally, this is a geometric object which is locally a quotient of a scheme by a finite group, but the geometric object remembers something about this local description. If the scheme is smooth, then we obtain the algebraic-geometric equivalent of an orbifold. More generally, an Artin (or algebraic) stack allows quotients by much more complicated equivalence relations. For example, the trivial action of an algebraic group $G$ on the point has a well-defined quotient in the world of algebraic stacks, and this quotient plays the role of the classifying space $BG$ in algebraic geometry. See [2] for a very brief survey, and [3] for a longer survey.

This essay would involve internalizing the (very complicated) definition of stacks, and giving some application(s). The most obvious application is the construction of the moduli space of stable curves [1]. Other possibilities include the construction of the Chow group for Artin stacks [5], and Artin’s criterion for algebraicity of stacks [4]. The former will require delving into the theory of algebraic cycles, the latter into deformation theory.

There are several sources for the definitions. The original papers [1] and [4] give concise definitions, and [6] covers these in a more expansive way (but is in french). There are a number of online resources, (follow the links from the wikipedia page on stacks) and the Stacks Project [8], the latter being a vast compendium of most of algebraic geometry and probably not so useful for a beginner. There is also a good new book on the subject by Martin Olsson, [7].

Relevant Courses

Necessary: Part III Algebraic Geometry (Michaelmas term).

References

13. Tropical Geometry .............................................
Professor M. Gross

Tropical geometry is algebraic geometry over the so-called tropical semiring, the semiring of
real numbers with addition being maximum and multiplication being ordinary addition. Thus
a “tropical polynomial” in \( n \) variables is really a function given as a maximum of a collection
of affine linear functions with integral slopes. The “zero locus” of such a function is interpreted
as the locus where such a function isn’t linear. For example, we define a tropical hypersurface
in \( \mathbb{R}^n \) as the non-linear locus of such a function.

While these resulting objects are very combinatorial in nature, there turns out to be a rich
and surprising relationship between tropical geometry and complex geometry. For example,
Mikhalkin [1] really started the subject by showing that curves in the complex projective plane
can be counted tropically.

There is now a wide literature in the subject, and some of this literature does not require very
much background in algebraic geometry. Thus this essay should be accessible to students who
have not taken the Part III Algebraic Geometry class, as long as they are willing to learn a
little bit of algebraic geometry on the way.

Chapter 1 of [2] gives an introduction to tropical geometry, and Chapter 4 gives the most
technologically advanced proof of Mikhalkin’s result. [3] gives a very elementary introduction
to the subject, and [4] gives an elementary, completely combinatorial proof (but see Chapter 2 of
[2] for the necessary background). Depending on the tastes of the essay writer, various aspects of
the theory can be explored. Possibilities include (a) the relationship between tropical geometry
and amoebas [5]; (b) enumerative applications [1], [4]; (c) applications to mirror symmetry,
especially [2], Chapter 5.

Relevant Courses

Essential: None
Useful: Part III Algebraic Geometry (Michaelmas term).

References


[2] Mark Gross, Tropical geometry and mirror symmetry, CBMS Regional Conference Series in


14. Classifying Toposes .............................................
Professor J. M. E. Hyland

Any Grothendieck Topos can be regarded as the classifying topos for some (by no means unique)
geometric theory. There is an early account of the basic ideas in [4] and a somewhat later one
from a different point of view in [3]. Peter Johnstone’s treatment in [2] is more sophisticated, placing classifying toposes within a general development of Topos Theory. In all three references classifying toposes occur quite late and as that suggests a good deal of both categorical and logical background is required for a full appreciation of the ideas. However those taking both the courses in Category Theory and Model Theory could certainly contemplate an essay. The task is made a little less intimidating by the existence of [1], written by Johnstone’s student, Olivia Caramello. That gets pretty directly to the subject though the background assumed is more subtle than may at first appear.

An essay could approach the topic from a number of different directions. One the one hand there is a good deal of fundamental theory which could be spelt out. Then again one could focus on ways of showing that particular toposes (however given) classify specific theories. (Examples of classifying toposes can be found in the references but one might well look for others.) More ambitiously the theory can be developed relative to a more general base topos; or one might try to treat recent developments regarding the Completeness Theorem. Anyone contemplating an essay on the topic is advised to discuss possibilities at an early stage.

Relevant Courses

Essential: Category Theory

Useful: Model Theory

References


15. Locally Presentable and Accessible Categories

Professor P. T. Johnstone

Locally presentable categories were introduced by Gabriel and Ulmer [1,2], and were an early attempt to capture the essential categorical structure of the category of models of a theory. The fact that they succeeded in doing just this, for a particular (very natural) class of ‘essentially algebraic’ theories, was proved by M. Coste [3]. More recently, attention has focused on the much larger class of accessible categories [4,5], which are categories of models of theories in a much broader sense; locally presentable categories are precisely those accessible categories which are complete as categories. An essay on this topic could either take as its goal the main theorem characterizing accessible categories as categories of models, or it could survey the way in which particular properties of the axiomatization of a theory are reflected in properties of its category of models. (Some examples of the latter may be found in [6].)

Relevant Courses

Essential: Category Theory
References

[1] F. Ulmer, Locally $\alpha$-presentable and locally $\alpha$-generated categories, in Reports of the Midwest Category Seminar V, Lecture Notes in Math. vol. 195 (Springer–Verlag, 1971), 230–247. (This is a summary in English of the main results of [2].)


16. Synthetic Differential Geometry .................................................. Professor P. T. Johnstone

In 1967, F.W. Lawvere suggested that the traditional analytic approach to differential geometry might be replaced by a ‘synthetic’ approach, in which one would begin by directly axiomatizing (a category containing) the category of smooth manifolds. Lawvere’s axioms are incompatible with classical logic, and thus with the traditional conception of what a smooth manifold is: it was not until the development of elementary topos theory in the 1970s that it became possible to give explicit models for them. An essay on this topic could either concentrate on developing the axiomatics (for which Anders Kock’s first book [1] is probably still the best introduction, although Kock’s later book [2] and René Lavendhomme’s [3] are also recommendable); or, more ambitiously, it could describe the construction of a ‘well-adapted’ model of the axioms, in which the classical category of manifolds is nicely embedded. (The latter would require the development of a good deal of topos theory; suitable references would include [4] and [5].)

Relevant Courses

Essential: Category Theory

References


17. Lagrangian Tori in $\mathbb{R}^6$ .................................................................

Dr A. M. Keating

Lagrangian submanifolds are distinguished half-dimensional submanifolds of symplectic manifolds – for instance, in the case of $\mathbb{C}^n$, the tori $\{ (z_1, \ldots, z_n) : |z_i| = a_i \}$, for some positive constants $a_i$. The nicest condition that one can impose on a closed Lagrangian submanifold in $\mathbb{C}^n$ is for it to be monotone; in the aforementioned example, this amounts to requiring that all of the $a_i$ be equal. Up to suitable notions of equivalence, there is a unique (automatically monotone) Lagrangian circle in $\mathbb{C}$. In $\mathbb{C}^2$, it is widely expected that there are two. The goal of this essay is to give an account of a beautiful result of Auroux, who, in contrast, produced an infinite collection of monotone Lagrangian tori in $\mathbb{C}^3$.

This essay should readily build on parts of the Symplectic Geometry course. After briefly recalling relevant definitions from the course, the essay should start by explaining Auroux’ construction from [1]; you may find it helpful to understand the perspective of Section 5 of [1], which draws on constructions in [2]. To tell the different tori apart, Auroux uses an invariant which comes from counting certain pseudo-holomorphic discs; the essay should proceed to give an account of this. You may choose to treat various amounts of Floer-theoretic background as a ‘black-box’.

Relevant Courses

*Essential:* Differential Geometry; Symplectic Geometry; basic notions from Algebraic Topology

*Useful:* Algebraic Geometry

References


For the symplectic geometry background (to be covered in the Lent term course):


18. Complements of Hyperplane Arrangements ........................................

Dr A. M. Keating

A hyperplane arrangement is a finite collection of affine hyperplanes in $\mathbb{C}^n$. These have been the object of considerable research, notably regarding the topological properties of their complements in $\mathbb{C}^n$. The goal of this essay is to study some of these properties. It should begin by discussing the fundamental group of the complement of a hyperplane arrangement, with starting point the Zariski–Van Kempen theorem. Several directions are then possible, for instance: Hattori’s result on the topology of the complement of a generic arrangement; Deligne’s proof that a simplicial arrangement gives a $K(\pi, 1)$ Eilenberg-MacLane space; the description of the cohomology ring of the complement in terms of generators and relations.
19. Yau’s Solution of the Calabi Conjecture .................................

Dr A. G. Kovalev

The subject area of this essay is compact Kähler manifolds. Very informally, a Kähler manifold is a complex manifold admitting a metric and a symplectic form, both nicely compatible with the complex structure. The Ricci curvature of a Kähler manifold may be equivalently expressed as a differential form which is necessarily closed. Furthermore, the cohomology class defined by this form depends only on the complex manifold, but not on the choice of Kähler metric. The Calabi conjecture determines which differential forms on a compact complex manifold can be realized by Ricci forms of some Kähler metric. Substantial progress on the conjecture was made by Aubin and it was eventually proved by Yau. This result gives, among other things, a powerful way to find many examples of Ricci-flat manifolds. The essay could discuss aspects of the proof and possibly consider some applications and examples. Interested candidates are welcome to contact A.G.Kovalev@dpmms for further details.

Relevant Courses

Essential: Differential Geometry, Complex Manifolds
Useful: Algebraic Topology, Elliptic Partial Differential Equations

References


29
20. Dirac Operators ......................................................... Dr A. G. Kovalev

The Dirac operator, for smooth functions from \( \mathbb{R}^n \) to \( \mathbb{C}^N \), may be defined as a first order differential operator whose square is the Laplacian. (Thus the simplest example of Dirac operator would be the usual derivative of complex-valued functions on \( \mathbb{R} \).) Unlike the Laplacian, which is well-defined on every oriented Riemannian manifold, the construction of Dirac operator requires the existence of a certain vector bundle, called the spinor bundle, over the base manifold. The essay could begin by explaining the significance of spinor bundles (cf. [1]), and why a Dirac operator can always be constructed when the dimension of the base manifold is 3 or 4. The kernel of a Dirac operator arises in many geometric and topological applications, including the Riemannian holonomy, deformations of volume-minimizing submanifolds, invariants of smooth 4-dimensional manifolds. The essay has an option to consider some of these topics. Interested candidates are welcome to contact A.G.Kovalev@dpmms and discuss the possibilities. The first two or three sections in [2] would be a good introductory reading (and a source of useful exercises!).

Relevant Courses

**Essential:** Differential Geometry, Algebraic Topology

**Useful:** Complex Manifolds

References


21. (No) Wandering Domains .................................................. Dr H. Krieger

A holomorphic self-map \( f \) of the Riemann sphere can have stable regions - known as *Fatou components* - where the long-term behaviour of points under iteration is predictable. Sullivan’s celebrated No Wandering Domains theorem [5] establishes that these components do not wander: that is, if \( U \) is a Fatou component of \( f \), then the set \( \{ U, f(U), f^2(U), f^3(U), \ldots \} \) is a finite collection of components.

In this essay, you will learn the basic theory of complex dynamics in one variable [4], and apply it to understand the proof of Sullivan’s theorem. You can then proceed in a number of directions: (1) wandering domains in transcendental dynamics [2], (2) wandering domains in higher-dimensional complex dynamics [1], or (3) no wandering domains in \( p \)-adic dynamics [3]. In each case, you will first develop the basic dynamical theory for the relevant setting.
Relevant Courses

**Essential:** Part II Riemann Surfaces, Differential Geometry (Part II or Part III).

**Useful:** Algebraic Geometry (for direction (2)), Number Fields / Theory (for direction (3)).

References


22. Effective Diophantine Approximation and Unlikely Intersections

Dr H. Krieger

The complex plane $\mathbb{C}$ parametrizes isomorphism classes of elliptic curves via the $j$-invariant. The principle of unlikely intersections predicts that a curve $f(x,y) = 0$ in $\mathbb{C}^2$ with no modular component should contain only finitely many points for which both coordinates are the $j$-invariant of an elliptic curve which admits an additional structure (known as complex multiplication). This finiteness was established by André in 1998, but his proof was ineffective; that is, it did not provide for a given curve any way to find all points on the curve with this property. In 2012 an effective version was proved independently by Kühne [5] and Bilu-Masser-Zannier [2], using the theory of linear forms in logarithms. This is a special case of what is known as the effective André-Oort conjecture.

The main goal of this essay will be to understand the theory of Weil heights in arithmetic geometry (see [4]) and the technique of linear forms in logarithms (see [3]), and to explain how they are used to provide effective bounds for questions of unlikely intersections as discussed above. An interested student might then proceed to related questions of the arithmetic geometry of the complex plane as moduli space of elliptic curves such as [1], or other instances of unlikely intersections (see [6]), or further results in effective Diophantine geometry (see [3]).

Relevant Courses

**Essential:** Part II Number Fields.

**Useful:** Part III Algebraic Number Theory, Elliptic Curves, and Algebraic Geometry.

References

23. Independence Results for Basic Axioms of Set Theory

Dr B. Löwe

The basic axiom systems of set theory are combined from the axioms (and axiom schemes) of Extensionality (Ext), Pairing (Pair), Union (Un), Power Set (Pow), Separation (Sep), Infinity (Inf), Replacement (Repl), and Regularity (Reg). Finite Set Theory (FST) is Ext + Pair + Un + Pow + Sep, Zermelo Set Theory (Z) is FST + Inf, Zermelo-Fraenkel Set Theory without Foundation (ZF₀) is Z + Repl, and Zermelo-Fraenkel Set Theory (ZF) is ZF₀ + Reg.

As a consequence, there are 2⁵ = 32, 2⁶ = 64, 2⁷ = 128, or 2⁸ = 256 subsystems of FST, Z, ZF₀, and ZF given by these axioms, respectively. The standard literature usually discusses only very few cases, usually just that

\[ \text{FST} < \text{Z} < \text{ZF}_0 < \text{ZF}, \]

where \( S < T \) means that \( T \) implies \( S \), but there is a structure satisfying \( S \), but not \( T \) [1, Chapter IV].

The goal of this essay is to explore the lattice of these subsystems in terms of finding structures that show that two of the subsystems are not logically equivalent.

As a first step, one would consider the subsystems of FST. When moving to subsystems of Z, one needs to discuss the Axiom of Infinity in more detail (since in its usual formulation, it needs a number of other axioms to hold).

If there is time for further exploration, one could consider the work by Mathias on weak systems of set theory, in particular, models of Z [2].

Relevant Courses

Essential: Part II Logic and Set Theory (or equivalent).
Useful: Part III Topics in Set Theory (or equivalent).

References

24. The Generalised Real Numbers

Generalised analysis is dealing with analytical, metric, and topological properties of generalisations of the set of real numbers to uncountable cardinals $\kappa$. In recent years, there has been a large number of new results in this field and a list of open problems was published in 2016 [4]. A fundamental problem in this field was the search for the correct analogue of the real number line for uncountable cardinals $\kappa$. In [3], Lorenzo Galeotti provided an answer to this question by defining the $\kappa$-reals based on Conway’s surreal numbers [2].

This essay aims at developing the basic theory of Galeotti’s $\kappa$-reals and studying some of their properties that distinguish them from the classical real numbers as well as other candidates for the generalised reals (such as Sikorski’s long reals). A useful reference is [1].

Relevant Courses

Essential: Part IB Metric and Topological Spaces (or equivalent), Part II Logic and Set Theory (or equivalent)

Useful: Part III Topics in Set Theory (or equivalent).

References


25. Long Blackwell Games

In the theory of infinite games, there is usually a trade-off between the (transfinite) length of the game and the size of the possible set of moves. E.g., if $\text{AD}_X[\alpha]$ denotes the axiom of determinacy for games of length $\alpha$ with moves in $X$, then Kechris proved that $\text{AD}_\mathbb{R}[\omega^2]$ is equivalent to $\text{AD}_\mathbb{R}[\omega]$.

Blackwell’s infinite games with slightly imperfect information [1] are not easily extended to transfinite length. Based on an idea by de Kloet [2, Section 2.4], transfinite Blackwell games for some ordinals were defined in [3, Section 9.2].

The goal of this essay is to develop the basic theory of long Blackwell games along the definitions from [3, Section 9.2], study Kechris’s theorem connecting the length of games and the size of the set of possible moves, and link them.
Relevant Courses

**Essential:** Part II Logic and Set Theory (or equivalent), Part II Probability and Measure (or equivalent)

**Useful:** Part III Topics in Set Theory (or equivalent)

References


26. Group Cohomology from the Topological Viewpoint ..........................  
Dr O. Randal-Williams

Group cohomology attaches to each group $G$ and commutative ring $k$ a graded-commutative $k$-algebra $H^*(G;k)$, which one may view as an attempt to “linearise” the group $G$. This may be defined and studied purely algebraically, but it is extremely profitable to instead consider it as the singular cohomology of a certain topological space $BG$ associated to the group $G$, so that one may use techniques from algebraic topology to study it: this point of view reveals it as a special case of so-called equivariant cohomology.

In this essay you should first give an introduction to this subject, discussing basic results such as the finite-generation of cohomology of compact Lie groups and the Localisation Theorem for equivariant cohomology. You should then explain Quillen’s theorem [2] relating the ring-theory of $H^*(G;\mathbb{F}_p)$ (its Krull dimension) to the group-theory of $G$ (the rank of its maximal elementary abelian $p$-subgroup), and generalisations. Finally, you should explain Symonds’ recent proof [3] of Benson’s Regularity Conjecture, which establishes a strong ring-theoretic property (Castelnuovo–Mumford regularity equals zero) for the cohomology of any finite group.

Along the way you will need to learn some pieces of Algebraic Topology, such as spectral sequences and characteristic classes.

Relevant Courses

**Essential:** Algebraic Topology

**Useful:** Algebras, or a willingness to pick up a certain amount of commutative algebra as you go. Algebraic Geometry, or a willingness to learn a bit about sheaves.
A surface bundle $\pi : E \to B$ is the analogue of a vector bundle in which vector spaces are replaced by surfaces. The study of surface bundles arises—directly or indirectly—in topology, group theory, differential and algebraic geometry, and arithmetic, and this multitude of perspectives makes it an especially interesting subject. This essay will focus on the algebraic topology of surface bundles, and especially on the characteristic classes $\kappa_i$ of surface bundles introduced by Miller, Morita, and Mumford.

After explaining how to construct the $\kappa_i$, you should show that they are not zero and indeed are algebraically independent in rational cohomology as the genus of the surface tends to infinity. This might be done following Miller’s generalisation [2] of a construction of Atiyah, or else by a surprising argument of Akita–Kawazumi–Uemura [1] using cyclic group actions on surfaces.

There are many directions to go after this, which can be discussed with me. One choice would be to explain the existence of algebraic relations among the $\kappa_i$ when one does not pass to the infinite genus limit, perhaps starting from Morita’s relations [3] constructed using the Abel–Jacobi map.

Relevant Courses

*Essential:* Part III Algebraic Topology, Part II Algebraic Geometry and Part II Riemann Surfaces.

References


28. $E_k$-algebras

Dr O. Randal-Williams

In homotopy theory many equations from classical algebra (e.g. being commutative: $a \cdot b = b \cdot a$) must be replaced by homotopies (e.g. a homotopy from $(a, b) \mapsto a \cdot b$ to $(a, b) \mapsto b \cdot a$). This additional data might also be subject to constraints, but in the same spirit these should not be in the form of equations, but rather as further homotopies. One may decide how far along this process to go, giving a hierarchy of structures

$$E_1 \supset E_2 \supset E_3 \supset \cdots \supset E_\infty$$

interpolating between the homotopical analogue of associative ($E_1$) and of fully commutative ($E_\infty$). While these extreme notions are the most common, the intermediate types of commutativity, especially $E_2$, play important roles.

This essay will explore the classical theory of $E_k$-algebras [4] and their relation to $k$-fold loop spaces, homology operations for $E_k$-algebras [1], and then give an exposition of some recent results [2, 3] applying $E_k$-algebras to study homological stability. It is quite open-ended and you should discuss your plans in detail with me first.

Relevant Courses

Essential: Part III Algebraic Topology.

References


29. The Heegaard Floer Contact Invariant

Professor J. A. Rasmussen

A contact structure $\xi$ on a 3-manifold $Y$ is a 2-dimensional sub-bundle of the tangent bundle which is maximally nonintegrable, in the sense that if $\xi = \ker d\alpha$ for some $\alpha \in \omega^1(Y)$, then $\alpha \wedge d\alpha \neq 0$. Contact structures arise naturally as the boundary of symplectic four-manifolds. Heegaard Floer homology is a package of 3-manifold invariants defined by Ozsváth and Szabó. One part of this package is an invariant of contact 3-manifolds $c(\xi)$.

The essay should define this invariant and describe some of its properties, including the fact that it is nonvanishing for fillable contact structures and 0 for overtwisted structures. This will require some discussion of Lefshetz fibrations and the Giroux correspondence. The basic properties of Heegaard Floer homology can be treated as a black box. Time and space permitting, you may want to discuss an application/extension of the invariant. Possible choices include the
construction of tight contact structures on Seifert fibred spaces, the construction of a tight contact structure with \(c(\xi) = 0\), or the extension of the contact invariant to sutured Floer homology.

**Relevant Courses**

*Essential*: Algebraic Topology, Differential Geometry

*Useful*: 3-Manifolds, Symplectic Topology

**References**


30. **Annular Khovanov Homology** .........................................................

Professor J. A. Rasmussen

Khovanov homology is an invariant of links in \(R^3\). It admits a more or less elementary combinatorial description, but has ties to many rich and interesting geometric theories. Annular Khovanov homology is a related invariant of links in the solid torus \(S^1 \times D^2\). The main goal of this essay is to understand a theorem of Grigsby, Licata, and Wehrli which says that annular Khovanov homology of a link \(L\) is naturally a representation of the Lie algebra \(\mathfrak{sl}(2)\).

The essay should explain the definition of the annular Khovanov homology and discuss Grigsby Licata and Wehrli’s construction in detail. It should then go on to discuss a subject of your choice related to annular Khovanov homology. Possible topics include the relationship between annular Khovanov homology and sutured Floer homology (due to Grigsby and Wehrli), Khovanov’s categorification of the Burau representation, or Queffelec and Rose’s construction of the annular \(\mathfrak{sl}(n)\) homology and the \(\mathfrak{sl}(n)\) action on it.

**Relevant Courses**

*Essential*: None

*Useful*: Lie Algebras and their Representations, Categorified Knot Invariants

**References**

31. **p-adic Modular Forms** .................................................

Dr G. Rosso

The main objective of the essay will be to study level 1 modular forms and congruences modulo $p$ among them; in particular you shall prove a structure theorem for modular forms modulo $p$, and how this can be used to define modular forms with $p$-adic weights.

If times allows, the problem of when a $p$-adic modular form is classical could be addressed.

**Relevant Courses**

*Essential:* Algebraic Number Theory, Elliptic Curves

*Useful:* Algebraic Geometry

**References**


32. **Symplectic Structures on Euclidean Space** .................................

Professor I. Smith

There are various constructions of non-standard “exotic” symplectic structures on Euclidean space; some are constructed by hand, but the most interesting arise from contractible affine varieties. Their exotic nature is often tied up with the existence of interesting Lagrangian submanifolds and the behaviour of dynamical systems on the manifold, as probed by holomorphic curve counting invariants (Floer theory, symplectic cohomology); there are many open questions about how much of the underlying affine algebraic geometry is captured by symplectic topology. This essay will construct some examples, prove they are exotic (taking some input from holomorphic curve theory as a black box where necessary), and discuss open questions.
Relevant Courses

**Essential:** Algebraic Topology, Differential Geometry, Symplectic Topology

**Useful:** Algebraic Geometry, Complex Manifolds, Topics in Floer theory

References


33. Packing Symplectic Tori .................................................................

Professor I. Smith

When can one fill the volume of a symplectic manifold by a collection of disjointly embedded standard symplectic balls of equal radius? (Or even one such ball?) Obstructions to such “packings” which go beyond volume constraints are bound up with many fundamental aspects of symplectic topology. One can relate configurations of embedded symplectic balls with symplectic forms on blow-ups of the given manifold, so symplectic packings are closely connected to finding the cone of cohomology classes containing symplectic forms on a blow-up. Recent work has led to a complete solution of the packing problem on symplectic tori with linear symplectic forms; the (ir)rationality of the form in cohomology plays a key role. This essay will outline the proof in the four-dimensional case, and perhaps say something in higher dimensions.

Relevant Courses

**Essential:** Algebraic Topology, Differential Geometry, Symplectic Topology, Complex Manifolds

**Useful:** Algebraic Geometry

References


34. Dynamics on K3 Surfaces ..................................................  
Professor I. Smith

K3 surfaces are all diffeomorphic to a smooth quartic surface in complex projective 3-space. They play a special role in the classification of complex surfaces, and have rich complex dynamics. The entropy of a holomorphic automorphism of a complex algebraic variety is given by the logarithm of its spectral radius for the action on cohomology, which means that dynamical questions can be approached lattice-theoretically. K3 surfaces admit automorphisms of positive entropy given by Salem numbers. Their construction makes extensive use of the Torelli theorem for K3 surfaces, Coxeter groups, and more. This essay will explain the Torelli theorem, discuss the Gromov-Yomdin theorem on topological entropy, construct some explicit positive entropy automorphisms, and discuss open questions.

Relevant Courses

Essential: Algebraic Topology, Differential Geometry, Complex Manifolds
Useful: Algebraic Geometry.

References


35. Expansion and Robust Expansion ........................................  
Professor A. G. Thomason

A graph is an expander if every set of vertices has a substantial sized neighbourhood; a typical definition might be that $|\Gamma(S)| \geq \lambda |S|$ for every set $S \subset V(G)$ with $|S| \leq |V(G)|/2$. Expansion has long been recognised to be a very useful property of graphs, both theoretically and in applications; it is easy to construct, say, hamiltonian cycles or large minors (subcontractions) in expanders, and expanding networks offer fast communication. More recently, a notion of robust expansion has emerged, allowing stronger constructions and leading to the proofs of not a few outstanding conjectures, such as Kelly’s, that every regular tournament decomposes into hamiltonian cycles. An essay could outline the consequences of expansion and how robust expansion differs, giving some examples of each in detail.

Relevant Courses

Essential: None
Useful: Combinatorics
References


36. Sidorenko’s Conjecture ....................................................... Professor A. G. Thomason

Let $H$ be some fixed graph. What is the minimum number of copies of $H$ that appear in a large graph $G$ of density $p$? Even if $H$ is $K_3$ this is a very difficult question, which was answered only recently. But, in the case that $H$ is bipartite, Sidorenko made the remarkable conjecture that the minimum is achieved by a random (or random-like) graph $G$. The conjecture has been proved for some $H$; curiously, it’s known for sparse $H$ such as trees, and for dense $H$, such as complete bipartite, but not for $H$ in between. An essay would look at the conjecture, some elementary arguments, and more recent arguments using entropy or dependent random choice.

Relevant Courses

Essential: Combinatorics

References


37. Cohomology of Number Fields ............................................. Professor J. A. Thorne

Group cohomology assigns to any group $G$ and $\mathbb{Z}[G]$-module $M$ a series of abelian groups $H^i(G, M)$. When $G = \text{Gal}(L/K)$ is the Galois group of a field extension and $M$ is a module of arithmetic interest (for example, $M = L^\times$), these groups have arithmetic meaning, and are commonly referred to as Galois cohomology groups. When $K$ is a number field and $G$ is the absolute Galois group of $K$, determination of the Galois cohomology of the units and of the idèle class group is essentially equivalent to class field theory.

The goal of this essay will be to prove the main theorems of global class field theory using Galois cohomology. A good essay will go further. One possible direction would be to discuss the Poitou–Tate duality theorems in Galois cohomology. These duality theorems are analogous to Poincaré duality for manifolds in algebraic topology. Another possible direction would be to discuss the application of global class field theory to the reciprocity law for the power residue symbol, which generalises quadratic reciprocity.
Relevant Courses

Essential: Algebraic Number Theory
Useful: Elliptic Curves

References


38. Arithmetic Statistics of Elliptic Curves

Professor J. A. Thorne

Let $p$ be a prime. If $E$ is an elliptic curve over $\mathbb{Q}$, it has an associated $p$-Selmer group $\text{Sel}_p(E)$. This is an $\mathbb{F}_p$-vector space whose dimension gives an upper bound for the rank of $E$ (i.e. the dimension of $E(\mathbb{Q}) \otimes \mathbb{Q}$). In recent years many mathematicians have studied the following question: what is the distribution of the quantity $d_{p,E} = \dim_{\mathbb{F}_p} \text{Sel}_p(E)$ as the curve $E$ varies? What is the average value? What about other statistics, e.g. the average size of the $p$-Selmer group?

The goal of this essay will be to explore some of the heuristics and theorems that have recently appeared about this question. A good essay should aim to discuss both heuristics and theorems, perhaps starting with the work of Poonen–Rains (that formulates precise heuristics, that are amenable to generalization) and proceeding to the work of Swinnerton-Dyer (which proves a result about the distribution of $d_{2,E}$ in certain quadratic twist families). An alternative reference is the paper by Klagsbrun, Mazur, and Rubin, which proves similar results for an elliptic curve over an arbitrary number field. Both references use Markov chains to describe the variation of the Selmer group under quadratic twists.

Relevant Courses

Essential: Elliptic Curves
Useful: Algebraic Number Theory

References


39. Komlós’s Conjecture in Discrepancy Theory .............................. Dr P. P. Varjú

Let $n$ and $d$ be two integers and let $u_1, \ldots, u_n \in \mathbb{R}^d$ be a sequence of vectors with $\|u_j\|_2 \leq 1$ for all $j = 1, \ldots, n$. A conjecture of Komlós predicts the existence of an absolute constant $C$ (it is independent of both $n$ and $d$!) such that there are numbers $\varepsilon_j \in \{-1, 1\}$ with

$$\|\varepsilon_1 u_1 + \ldots + \varepsilon_n u_n\|_\infty \leq C.$$ 

To see where this is coming from, consider the standard basis of $\mathbb{R}^d$ in the role of $u_j$. The essay will discuss progress towards this remarkable conjecture.

Relevant Courses

No courses are required but basic knowledge of Probability is very useful for this essay.

References


40. Equidistribution of Roots of Polynomials .............................. Dr. P. P. Varjú

The essay will discuss the phenomenon that polynomials with small integer coefficients tend to have their roots accumulated near the unit circle and they are approximately evenly distributed there. Precise statements of this kind have been proved by Erdős and Turán [3] and Bilu [2]. These results have been revisited by many authors because of their importance to number theory.

Relevant Courses

No courses are required but basic knowledge of Galois theory and Fourier analysis is very useful for this essay.
41. Croot-Sisask Almost-Periodicity and Applications

Dr J. Wolf

Originally developed for the purpose of strengthening results on the existence of long arithmetic progressions in sum sets of subsets of the integers, the Croot-Sisask almost-periodicity technique has been instrumental in several recent breakthroughs: Sanders employed it to prove almost-logarithmic bounds in Roth’s theorem, and Schoen and Sisask used it to obtain essentially tight bounds in Roth’s theorem with four variables. It has thus become firmly established as an essential component of the toolkit of modern additive combinatorics.

The essay will motivate, describe and prove the original almost-periodicity result (there are by now several proofs available in the literature), and then go on to cover at least two of its more substantial applications. While the technique itself is purely combinatorial (or probabilistic, depending on one’s point of view), familiarity with the discrete Fourier transform is indispensable for making sense of the applications. A good essay will devote considerable attention to developing the necessary background on regular Bohr sets as substructures that are approximately closed under addition.

Relevant Courses

Essential: Introduction to Discrete Analysis.

Useful: Introduction to approximate groups.

References

42. Concentration and Functional Inequalities and their Relation to Markov Processes

Dr S. Andres

Consider a Markov process \((X_t)_{t \geq 0}\) and assume that \(X\) is stationary and ergodic, which ensures ‘convergence to equilibrium’, that is the convergence in law of \(X_t\) to the stationary distribution as \(t \to \infty\). However, the rate of convergence (which is of interest in many areas, for example, in non-equilibrium statistical mechanics or Markov chain Monte Carlo algorithms) is unknown in general.

It turns out that the convergence to equilibrium can be analysed by certain functional inequalities such as Poincaré inequalities, Sobolev inequalities or Logarithmic Sobolev inequalities, which lead to (precise) quantitative bounds on the convergence to equilibrium and other properties of the transition semigroup of \(X\) such as ultra- and hypercontractivity.

On the basis of the monograph [1] and the lecture notes [2], a successful essay will describe in detail the relation between those functional inequalities and semigroup properties, and will discuss applications by means of at least one example.

Relevant Courses

**Essential:** Markov Chains, Advanced Probability

**Useful:** Applied Probability, Linear Analysis or Functional Analysis

References


43. Piecewise Deterministic Markov Processes

Dr S. A. Bacallado

Piecewise deterministic Markov processes (PDMPs) evolve according to an ordinary differential equation for random lengths of time between stochastic jumps in the state space. They were introduced by Davis in 1984 [1] and have recently found use in Markov chain Monte Carlo (see [2,3] and references within). They have the desirable property that the deterministic pieces can travel long distances for a limited computational cost, avoiding the diffusive behaviour of many reversible Markov chain samplers.

A more applied essay could review the main aspects of the algorithms discussed in [2, 3] and illustrate their application in a simple statistical problem. A more theoretical essay could focus
on what can be proven about the mixing time of a PDMP, for example, through the analysis in the recent manuscript by Andrieu et al. [4].

**Relevant Courses**

*Essential:* None

*Useful:* Bayesian Modelling and Computation, Advanced Probability

**References**


**44. Random Matrix Eigenvalue Statistics**

Dr R. Bauerschmidt

Consider the *Gaussian Unitary Ensemble* (GUE): a complex hermitian $N \times N$ matrix with independent complex Gaussian entries above the diagonal. (The entries below the diagonal are then determined by the constraint that the matrix is hermitian.) The distribution of the $N$ real eigenvalues of the GUE can be computed explicitly. This distribution has interesting structure: it is a determinantal point process. This allows, for example, the limiting distribution of the gap between two neighbouring eigenvalues or the distribution of the largest eigenvalue to be computed explicitly. These distributions turn out to be non-Gaussian, yet ubiquitous — or universal. For example, the rescaled distribution of the gaps between two zeroes on the critical line of the Riemann zeta function is conjectured to converge to the same distribution as the eigenvalue gap distribution of the GUE.

The goal of this essay is to derive the joint eigenvalue distribution of the GUE, to derive its global limiting density (the Wigner semicircle law), and to derive the determinantal structure of the eigenvalue distribution and its limiting correlation kernel (the Sine kernel). Then either a derivation of the limiting eigenvalue gap distribution (the Gaudin distribution) as well as the distribution of the largest eigenvalue (the Tracy–Widom distribution) should be given, or an introduction to the universality problem for random matrices.

**Relevant Courses**

*Essential:* Advanced Probability, Stochastic Calculus and Applications
A random walk on the vertices of a graph is a particle which in each time step moves to a neighbor of its currently location with equal probability. The loop erasure of a random walk is obtained from by starting with a random walk and then chronologically erasing the loops made by the random walk. It was introduced as a toy model for the so-called self-avoiding walk (SAW) by Lawler. It was proved by Lawler-Schramm-Werner that if one considers loop erased random walk on $\mathbb{Z}^2$, then in the fine mesh limit it converges to the Schramm-Loewner evolution (SLE) with parameter $\kappa = 2$. In the same work, Lawler-Schramm-Werner showed that the peano curve associated with a uniformly random spanning tree converges to SLE$_8$.

A successful essay will review the proof of these results as well as discuss more recent developments on scaling limit results for loop-erased random walks.

References


now fix $p > p_c$ and consider the unique infinite cluster $C$ of bond percolation with parameter $p$. One way of probing the geometry of $C$ is to perform a simple random walk on it, which is a process that at each time step jumps to a uniformly chosen neighbor (in the graph $C$) of its current position. Barlow obtained Gaussian upper and lower bounds on the transition density for the continuous time walk and a few years later, Berger and Biskup and independently Matthieu and Piatnitski proved that for almost every percolation configuration the path of the walk suitably rescaled converges weakly to that of non-degenerate, isotropic Brownian motion. A successful essay should give an account of these developments and include proofs (or overviews of proofs) of the important results.

**Relevant Courses**

*Essential:* Advanced Probability, Percolation

**References**


47. Random Walks on Height Functions

Professor J. R. Norris

According to Donsker’s invariance principle, any zero-mean, finite-variance random walk on the integers converges weakly under diffusive scaling to a Brownian motion. The diffusivity of the limit Brownian motion is simply the one-step variance of the random walk. The essay will examine the phenomenon of convergence to Brownian motion in a more general setting.

Suppose we are given a finite bipartite graph $G$ with edge set $E$. Let us say that a function $f : G \to \mathbb{Z}$ is a height function if $|f(x) - f(y)| = 1$ whenever $(x,y) \in E$, and say that two height functions $f$ and $g$ are neighbours if $|f(x) - g(x)| = 1$ for all $x \in G$. Consider the random walk $(F_n)_{n \geq 0}$ on the set of height functions, that is, the random process which moves in each time step from its present state to a randomly chosen neighbour.

The aim of the essay is to show that the average height process $\bar{F}_n = \frac{1}{|G|} \sum_{x \in G} F_n(x)$ converges under diffusive scaling to a Brownian motion and to determine, at least in some special cases, the diffusivity of the limit. See Chapter 7 in [2] for an introduction to diffusion approximation. Ideas from [3] on correctors may also be useful. Some aspects of this essay may be open problems. Original work will receive special credit but is not necessary for an essay of Distinction standard.

**Relevant Courses**

*Essential:* None

*Useful:* Advanced Probability
Brownian motion on a Riemannian manifold is the unique random process which satisfies the two conditions that it is a martingale and that its quadratic variation is given by the metric tensor. Properties of this process are then closely related to both local and global properties of the manifold.

The essay will present an account of one or more constructions of Brownian motion on a Riemannian manifold and will discuss ways to characterize Brownian motion in terms of discrete approximations, as a Markov process, using stochastic differential equations, and via the heat equation. Then some further topics can be chosen in which the behaviour of Brownian motion is analysed. Examples of such topics are: recurrence and transience, behaviour under projections, Brownian bridge and geodesics, long-time behaviour, the case of Lie groups.

The nature of this essay is a synthesis of material in a well developed field. Given the availability of many relevant sources, special credit will be given for an attractive and coherent account.

Relevant Courses

Essential: None
Useful: Advanced Probability, Stochastic Calculus and Applications, Differential Geometry

References

49.  Optimal Allocation in Sequential Multi-armed Clinical Trials with a Binary Response  

Dr D. Robertson and Dr S. S. Villar

Before a novel treatment is made available to the wider public, clinical trials are undertaken to show that the treatment is safe and efficacious. Multi-armed trials, i.e. trials in which multiple treatment options are simultaneously considered, are increasingly being used to speed up the drug development process [1]. In such trials, sequential changes to the allocation probabilities for the different treatments can be made based on the accrued data, in order to achieve certain objectives. Efficiency and ethical goals create trade-offs that are well studied in sequential two-armed trials [2]. In the multi-armed setting, solutions that preserve statistical power while optimising an ethical measure are complex [3, 4, 5] and far less studied.

This essay could focus on cases in which optimal allocation admits a closed-from solution, explaining their derivation and discussing the limitations of the required assumptions. Alternatively, the focus could be on the numerical implementation of optimisation algorithms to solve cases in which a closed-formed expression is intractable, and analysing the properties of the resulting optimal allocation designs through computer simulations. The possibility of exploring different optimisation criteria and their solutions is encouraged.

Relevant Courses

Essential: None
Useful: Statistics in Medical Practice, Topics in Convex Optimisation

References


50. The EM and $k$-means Algorithms  

Dr M. Gataric, Dr J. Jankova and Professor R. J. Samworth

Clustering, a canonical example of unsupervised learning, is one of the most common tasks of exploratory data analysis, with applications in a huge variety of scientific domains, including machine learning, image analysis, bioinformatics and many others (e.g. [1], Chapter 9). Two classical algorithms for this task are the Expectation–Maximisation (EM) and $k$-means algorithms ([2], [3]). Despite their enormous popularity, however, historical attempts to explain their behaviour have provided only limited insights ([4], [5], [6]), and, at least until very recently, both methods have been shrouded in mystery in terms of understanding when and why they
work. Many of the difficulties stem from the fact that the underlying optimisation problems these algorithms attempt to solve are non-convex. However, in the last couple of years, there has been considerable progress towards understanding these algorithms ([7], [8], [9]).

After a brief historical review and description of the types of problem to which these algorithms can be applied, this essay would focus on these recent developments in understanding the EM and \( k \)-means algorithms. There is considerable scope for the candidate to explore their performance in new models, either theoretically or empirically.

**Relevant Courses**

*Essential:* None

*Useful:* Bayesian Modelling and Computation, Modern Statistical Methods

**References**


### 51. Applications of Random Matrix Theory in Statistics

**Professor R. J. Samworth and Dr Z. Zhu**

Large random matrices occur in many areas of modern Statistics. Examples include estimators of high-dimensional covariance matrices and their inverses (e.g. [1], [2]), sparse Principal Components Analysis ([3], [4]) and random projections ([5], [6], [7]), among many others. Concentration inequalities and spectral properties often lie at the heart of their analysis, and results on rates of convergence etc. may be heavily dependent on the choice of matrix norm.

This essay will need to cover some of the relevant background in random matrix theory (e.g. [8], [9]), but should focus on how these results are applied in statistical contexts.
Relevant Courses

Essential: None
Useful: Modern Statistical Methods, Topics in Statistical Theory

References


52. Estimation of Heterogeneous Treatment Effects

Dr T. B. Berrett and Professor R. J. Samworth

In the potential outcomes model for causal inference ([1], [2]), we consider independent copies of quadruples \((X, A, Y^0, Y^1)\), where \(X\) is a covariate taking values in \(\mathbb{R}^d\), \(A\) is a treatment indicator taking values in \(\{0, 1\}\), and \(Y^a\) is the observed response when \(A = a\) (we do not observe \(Y^{1-a}\)). The main quantity of interest is the so-called heterogeneous treatment effect ([3], [4]), given by

\[
\tau(x) := E(Y^1|X = x) - E(Y^0|X = x).
\]

The main difficulty here is that we observe only one of the two potential outcomes \(Y^0\) and \(Y^1\), so estimation of \(\tau(\cdot)\) is impossible without further assumptions. A standard approach is to assume that there are no unmeasured confounders, i.e. \((Y^0, Y^1)\) and \(A\) are conditionally independent given \(X\). This facilitates estimation, e.g. based on propensity weighting ([5]).

One may also consider estimation of integral functionals of \(\tau(\cdot)\), for instance using higher-order influence functions ([6]). These ideas rely on restrictive assumptions on the support of the covariate being compact, however, and appropriate functions being bounded away from their extremes. Such restrictions could possibly be alleviated using recent developments in the theory of functional estimation ([7], [8]), and this may provide an interesting new direction for an ambitious candidate.
Relevant Courses

Essential: None
Useful: Modern Statistical Methods, Topics in Statistical Theory

References


53. Recent Developments in False Discovery Rate Control

Dr R. D. Shah

Since its introduction in 1995, Benjamini and Hochberg’s paper [1] introducing the False Discovery Rate (FDR) has now become perhaps the most cited statistics paper of the last 25 years. Whilst the method they propose for performing FDR control continues to be hugely popular, in recent years there have been a number of important developments that improve upon this in different settings, and multiple testing is currently a highly active area within methodological statistics.

One body of work has considered various forms of structure among the hypotheses [2–4]. For example, it could be the case that one hypothesis being false logically implies that another may be false. Another interesting line of work has looked at procedures that work with certain test statistics rather than simply thresholding p-values at an appropriate point ([5–7]).

This essay could review some of the innovations in multiple testing that have been introduced in recent years and compare them theoretically and / or empirically. Another option would be to combine or slightly modify existing procedures or ideas with the aim of obtaining improved performance in certain settings.

Relevant Courses

Essential: Modern Statistical Methods

53
References


54. Statistical Inference Using Machine Learning Methods .................

Dr R. D. Shah

The field of machine learning has much overlap with statistics, but has been predominantly concerned with prediction problems. Here it has had great successes with random forests, boosted trees, kernel machines and neural networks, among others, enjoying spectacular predictive performance in a variety of applications.

Statisticians, on the other hand, are often also concerned with parameter estimation for particular models, and uncertainty quantification. Recently, there has been a string of work aiming to harness the predictive power of machine learning methods for these more statistical goals. These can often lead to hypothesis tests with greater power than would be achievable using more classical tools, for example, or much more robust procedures whose validity holds across a far greater range of data-generating processes. The debiased Lasso [1] may be viewed as one example of this, where the Lasso is used to build confidence intervals for regression coefficients in high-dimensional settings. Other recent work considers statistical estimators based on random forests [2], deep neural networks [3] and regression splines [4]. The papers [5–7] propose more general procedures where prediction methods are plugged in, and study their properties.

Much of the work is tightly connected to the rich field of semiparametric statistics, and one option (among several) for the essay would be to review some of the papers referenced below and set some of the work within this context. Another option would be to focus more closely on a smaller subset of the papers and study some examples of the general methodology presented or propose some extensions.

Relevant Courses

*Essential:* Modern Statistical Methods

*Useful:* Topics in Statistical Theory, Statistical Learning in Practice
References


55. Model-Free No-Arbitrage Bounds ..................................................

Dr M. Tehranchi

An important problem in financial risk management is to some how incorporate the information contained in quoted asset prices to find prices and hedging strategies for contingent claims. A classical approach is to introduce a model which is consistent with the observed prices, and then compute the prices and hedging strategies as predicted by the model. Unfortunately, more than one model may be consistent with a given set of data, and hence over-reliance on the predictions of a model exposes the practitioner to model-risk.

An alternative approach is compute upper lower bounds for contingent claim prices over all no-arbitrage models consistent with the data. Furthermore, it is often possible to compute the hedging strategies corresponding to the worst-case models.

When the observed market data consists of the price of a stock and a family of call option prices and the contingent claim is a barrier-style option on the stock, then the second problem can be reformulated in terms of a Skorokhod embedding problem. This technique has proved very effective. More recently, the developing theory of optimal martingale transport has provided a unified framework for studying the general model-free no-arbitrage pricing and hedging problem.

This essay should survey the recent literature on model-free pricing and hedging. Possible topics include the theory of Skorokhod embedding and the construction of certain optimal stopping times, the theory of optimal martingale transport, or a comparison between these two perspectives on the pricing and hedging problem.

Relevant Courses

*Essential:* Advanced Financial Models, Stochastic Calculus & Applications

*Useful:* Advanced Probability
References


56. Polynomial Preserving Processes

Dr M. Tehranchi

A real-valued Markov process $X$ is polynomial preserving if the function $u$ defined by

$$u(t,x) = \mathbb{E}[f(X_t)|X_0 = x]$$

is a polynomial in $x$ for all $t$ whenever $f$ is a polynomial. There is growing interest in modelling financial quantities with such processes since the computations involved in pricing certain derivative contracts are reasonably tractable.

This essay will survey the literature on polynomial preserving processes and related variants. Focus can be on the mathematical properties, such as characterisations of their generators, or can be on a particular application in finance, exploring their advantages and disadvantages compared to other modelling frameworks.

**Relevant Courses**

*Essential:* Advanced Financial Models, Stochastic Calculus & Applications

*Useful:* Advanced Probability

References


57. Precision Higgs Mass Predictions in Minimal Supersymmetry

Professor B. C. Allanach

The Minimal Supersymmetric Standard Model (MSSM) is still regarded by many to be an attractive TeV-scale extension to the Standard Model (however, in this essay, it is not necessary
to review supersymmetry or the MSSM at all). The prediction of a Higgs boson whose properties match those of the experimentally discovered particle is an obvious priority. The calculation of its mass in particular has rather large radiative corrections, and is calculated to a relatively high order in perturbation theory, in various different schemes and approximations. It is a subject of active research as to which scheme or approximation links the Higgs boson mass prediction most precisely to the rest of the model.

The purpose of this essay is to find out and present the issues in the precision Higgs mass prediction in the MSSM, while providing an overall context.

The first half of the essay will set the scene in terms of experimental data for the Higgs boson discovery, and some analytical predictions for the lightest CP even Higgs boson mass prediction in terms of the other parameters of the model. The second half should address the important issues coming from approximations and different schemes, along with current attempts to address them, and their short-comings.

**Relevant Courses**

*Essential*: Quantum Field Theory, Standard Model, Particles and Symmetries, Advanced Quantum Field Theory

**References**


*(and references therein)*

58. **Edge-Turbulence Interaction and the Generation of Sound**

Dr L. J. Ayton

An unavoidable source of noise occurs when hydrodynamic pressure fluctuations move over a surface and encounter sudden changes to that surface, resulting in the fluctuations refracting and scattering into acoustic waves. A simple example occurs when a turbulent boundary layer above an aerodynamic wing encounters the sharp trailing edge.

This essay will discuss early analytical models of turbulence-edge interaction noise [1-4] then present ideas for how to update these models to include more realistic effects, for example the inclusion of surface roughness which could occur due to the inevitable damage of surfaces over time [5].

**Relevant Courses**

*Essential*: None

*Useful*: Perturbation Methods
59. Extrema in Gaussian Random Fields as a Proxy for Galaxy Clustering

Dr T. Baldauf

In the past two decades our picture of the Universe has been dramatically refined by observational campaigns that led to the measurement of the temperature fluctuations in the Cosmic Microwave Background (CMB) and the discovery of the accelerated expansion using supernovae. These new insights led to new ideas about the origin and evolution of the Universe and raised a plethora of new questions. For instance, what are the processes that seeded the rich structures that we observe today and what is causing the Universe to accelerate?

Large Scale Structure (LSS), the distribution of matter and galaxies in the late time Universe has been shown to have the potential to answer some of these questions. For instance, LSS is able to put an upper bound on the total neutrino mass, it can rule out some modifications of gravity and constrain the properties of dark energy models. Extracting these signals from upcoming galaxy surveys is a non-trivial task: the observable galaxies, which form in virialized clumps of dark matter (haloes), are an imperfect tracer of the matter distribution.

In the simplest model the galaxy two correlation function $\xi_{gg}$ is assumed to be a linearly biased version of the (unobservable) matter correlation function $\xi_{mm} = b^2 \xi_{mm}$. This model ignores the finite size of the collapsed dark matter haloes and the specific locations of their formation sites. A simple, yet phenomenologically interesting extension to this model is to assume that dark matter haloes form from maxima in the initial Gaussian density field. This so called peak model [2] makes distinct predictions for the shape of the baryon acoustic oscillation feature in the correlation function, the motion of haloes and their stochasticity.

In this essay you will first understand how the rich structure in the Universe arises from small, Gaussian fluctuations [1]. You will then focus on the properties of maxima in the Gaussian initial conditions (Lagrangian space) and describe their clustering properties [2,3,4] and stochasticity [5]. Finally you will explore how these properties translate into the evolved late time galaxy distribution [6,7].

**Relevant Courses**

*Essential:* Cosmology

*Useful:* Advanced Cosmology, Quantum Field Theory
References


60. Dualities and the Equivalence of Physical Theories .................

Dr J. N. Butterfield

A duality in physics is like a ‘giant symmetry’. In short: while a symmetry maps a state of the system into an appropriately related state (namely, one with the same values for a salient set of physical quantities): in a duality, an entire theory is mapped into another appropriately related theory. The important cases are those in which:

(i): a strong coupling regime of the first theory is mapped onto a weak coupling regime of the second theory; so that a problem that is hard in the first can be addressed, maybe solved, by solving an easier problem in the second; and-or

(ii) the theories seem to be about very disparate systems and-or phenomena, which suggests there are some deep links underneath the apparent differences: (the currently most famous example of this being gauge/gravity duality).

There are several good recent philosophical discussions of both aspects (i) and (ii) (which of course overlap). For a sample that addresses (i), cf. the papers in [1]. For a sample that addresses (ii), cf. the papers in [2].

This essay is about (ii): specifically, how to make precise—and so assess as true or false—the idea that two dual theories are ‘the same theory’. So the essay calls for some general account of what a duality is. The philosophical literature has suggestions about this, with an eye on topic (ii). The articles tend to treat several examples, and to compare the differences between dual theories with different fixings of a gauge. A sample of papers is [3]. The essay also calls for some account of the individuation of physical theories, i.e. an account of ‘theory’ precise enough to underwrite judgments of sameness and difference. This is clearly a topic for which one looks to logic and philosophy to supply an account. Indeed, these disciplines have considered various proposals about this, wholly independent of considerations about dualities in physics: the topic is called ‘theoretical equivalence’. This tradition has recently been revived by appeal to category theory. For a sample of recent papers, cf. [4].

59
The aim of the essay is to assess the various proposals (a) about what a duality is, and (b) about the equivalence of theories, so as to get a considered verdict on whether two given dual theories are ‘the same theory’. Of course, the essay need not defend, or adopt, a single proposal (a). Similarly, it need not defend, or adopt, a single proposal (b). It can survey the different verdicts. (Nor is it required that the verdict about equivalence should be the same for any two dual theories.)

**Relevant Courses**

*Essential:* None

*Useful:* Philosophical aspects of symmetry and duality.

**References**


http://philsci-archive.pitt.edu/14663


http://philsci-archive.pitt.edu/11923/

61. Symmetry and Symplectic Reduction .................................
Dr J. N. Butterfield

Symplectic reduction is a large subject in both classical and quantum mechanics. The course, ‘Philosophical Aspects of Symmetry and Duality’, will give an introduction to the classical aspects. This introduction will start from Noether’s theorem in a classical Hamiltonian framework, and then explain (in terms of modern differential geometry) the ideas of: Lie group actions; the co-adjoint representation of a Lie group $G$ on the dual $g^*$ of its Lie algebra $g$; Poisson manifolds (a mild generalization of symplectic manifolds that arise when one quotients under a symmetry); conserved quantities as momentum maps. With these ideas one can state the main theorems about symplectic reduction. The course will focus on the Lie-Poisson reduction theorem, which concerns the case where the natural configuration space for a system is itself a Lie group $G$. This occurs both for the pivoted rigid body and for ideal fluids. For example, take the rigid body to be pivoted, so as to set aside translational motion. This will mean that the group $G$ of symmetries defining the quotienting procedure will be the rotation group $SO(3)$. But it will also mean that the body’s configuration space is given by $G = SO(3)$, since any configuration can be labelled by the rotation that obtains it from some reference-configuration. So in this example of symplectic reduction, the symmetry group acts on itself as the configuration space. Then the theorem says: the quotient of the natural phase space (the cotangent bundle on $G$) is a Poisson manifold isomorphic to the dual $g^*$ of $G$’s Lie algebra. That is: $T^*G/G \cong g^*$. There are several ‘cousin’ theorems, such as the Marsden-Weinstein-Meyer theorem. Main texts for this material include [1]. The course’s exposition will use [2].

The essay should, starting from this basis, expound one or other of the following two topics. (Taking on both would be too much.)

(A): The first topic is technical and concerns the application of these classical ideas to quantum theory: more specifically, the interplay between reduction and canonical quantization. Physically, this is a large and important subject, since it applies directly to some of our fundamental theories, such as electromagnetism and Yang-Mills theories. The essay can confine itself to the more general aspects: which are very well introduced and discussed by Landsman and Belot; [3].

(B): The second topic is more philosophical. It concerns the general question under what circumstances should we take a state and its symmetry-transform to represent the same state of affairs—so that quotienting under the action of the symmetry group gives a non-redundant representation of physical possibilities? This question can be (and has been) discussed in a wholly classical setting. Indeed, the prototype example is undoubtedly the question debated between Newton (through his ammanuensis Clarke) and Leibniz: namely—in modern parlance—whether one should take a solution of, say, Newtonian gravitation for $N$ point-particles and its transform under a Galilean transformation to represent the same state of affairs. This topic is introduced by the papers in [4]. In particular, Dewar discusses how, even when we are sure that a state and its symmetry-transform represent the same state of affairs, quotienting can have various disadvantages.
Relevant Courses

Essential: None
Useful: Symmetries, Fields and Particles; Philosophical Aspects of Symmetry and Duality.

References


62. Viscoelastic Instabilities in Soft Matter .......................... Professor M. E. Cates

In many non-Newtonian fluids, such as entangled polymer solutions or liquid crystals, nonlinear instabilities can arise in shear flow at zero Reynolds number (that is, in the absence of the inertial nonlinearity of the Navier Stokes equation) [1]. This is often because these materials have a nonlinear relation between viscoelastic stress and strain rate history, which replaces the linear relation between viscous stress and strain rate in a Newtonian fluid [2]. Instabilities include shear-banding – in which a homogeneously sheared fluid separates into zones of high and low viscosity [3]– and tumbling, in which a fluid with orientational order (such as a liquid crystal) undergoes periodic or chaotic dynamics of the axis of order [4]. The essay should give a synoptic survey of such viscoelastic instabilities before focusing in more depth on one or two of them, freely chosen.

62
63. Mixing Efficiency in Stratified Fluids .......................... Professor C. P. Caulfield

Stratified fluids (i.e. fluids with density differences) are ubiquitous in the environment and industry. A particularly important issue is the extent to which turbulent motions irreversibly change the density distribution within the flow (i.e. ‘mix’ the fluid). Since mixing in a stratified fluid inevitably changes the potential energy of the flow, it is of great interest to understand the efficiency of the mixing, i.e. the proportion of the work done on the flow that leads to irreversible mixing. This essay should investigate the underlying issues of the energetics of high Reynolds number stratified flows, and then consider some of the various approaches to describe and parameterize stratified mixing processes, many of which have quite surprising and counter-intuitive aspects.

References


Relevant Courses

Essential: None

Useful: Fluid Dynamics of the Environment, Hydrodynamic Stability

References

https://doi.org/10.1007/BF00366504
https://doi.org/10.1080/00018730601082029
https://arxiv.org/abs/cond-mat/0702047
https://doi.org/10.1039/b707980j
https://doi.org/10.1103/PhysRevE.66.040702
Stratified shear flows, where both the fluid density and the velocity vary with height, are extremely common both in the environment and in industrial contexts. It is of great practical importance to understand how such flows undergo the transition to turbulence, as turbulence typically hugely increases mixing, transport and dissipation within such flows. It is commonly believed that ‘normal’ mode flow instabilities play a central role in this transition process, and the conventional argument is that the ‘most unstable’ normal mode will dominate the nonlinear evolution of the flow, and hence lead the flow to transition. However, the underlying linearized operator is non-normal, and so it is possible for substantial transient growth of perturbations to occur. Although this has been widely studied in unstratified flows, [1,2] the transient behaviour of stratified flows has been much less-studied. Also, stratified shear flows are prone to multiple, qualitatively different primary and secondary instabilities (particularly when the density distribution develops sharp interfaces [3]) and it appears that the transition to turbulence is typically associated with secondary instabilities which only develop once the primary instability has saturated [4]. There are also several interesting mathematical issues about the ‘optimal’ measures of perturbation growth to use, as the potential energy as well as the kinetic energy of the perturbation varies in a stratified flow [5], and this essay could approach the general issue of perturbation growth in stratified shear flows from a variety of mathematical and computational directions.

Relevant Courses

Essential: Hydrodynamic Stability
Useful: Fluid Dynamics of the Environment

References

A fundamental problem in quantum information theory is to determine how well lost information can be reconstructed. Crucially, the corresponding recovery operation should perform well without the knowledge of the information to be reconstructed. A useful figure of merit for the performance of such recovery operations is given by an entropic quantity called the *quantum conditional mutual information* (QCMI). An explicit characterisation of tripartite quantum states for which the QCMI is zero, and which hence saturate the powerful strong subadditivity inequality for the von Neumann entropy, has been obtained in [1]. This characterisation also yields a necessary and sufficient condition for quantum error correction. A tripartite quantum state for which the QCMI is zero, is said to form a so-called (short) quantum Markov chain. It is hence natural to expect that tripartite quantum states whose QCMI is small but non-zero are close to a quantum Markov chain. However, counterexamples of this intuition were provided in [2], and the problem of characterising such states remained open for years. A crucial breakthrough on this problem was finally made by Fawzi and Renner in [3]. This was followed by further interesting developments (see e.g. [4], [5] and [6]) of the notions of recoverability of quantum information and approximate quantum Markov chains. This essay will entail a review of this very timely and interesting research area. Students are not expected to cover all the papers listed below.

**Relevant Courses**

*Essential*: Quantum Information Theory (Lent)

*Useful*: Markov Chains (Part IB)

**References**


The problem of maximizing a degree-four polynomial \( p \in \mathbb{R}[x_1, \ldots, x_n] \) on the unit sphere \( S^{n-1} = \{ x \in \mathbb{R} : \|x\|_2 = 1 \} \) is hard in general. A sequence of tractable convex relaxations can be used however to approximate the solution of such an optimization problem. More precisely there is a sequence \( v_1 \geq v_2 \geq \cdots \geq p^* = \max_{x \in S^{n-1}} p(x) \) such that each \( v_k \) can be computed as a semidefinite program of size \( n^k \), and where \( v_k \to \max p \). The quantity \( v_k \) is defined as the smallest \( \gamma \) such that \( \gamma - p \) is a sum-of-squares of polynomials of degree \( k \) modulo the sphere. The question of how fast the sequence \( (v_k) \) converges to \( \max p \) is still not very well understood.

The objective of this essay will be to summarize the known results related to this question, and to explore the connections between this problem and the problem of separability testing in quantum information theory.

**Relevant Courses**

*Essential*: Topics in Convex Optimisation

**References**


The tropical atmosphere is driven by radiative heating towards a state which is relatively warm at the surface and relatively cold at altitude and as a result strong convection develops, manifested by tall cumulus clouds. However the entire tropics is not in a state of active convection, but instead there is strong spatial variation at scales ranging from those of individual clouds to scales of hundreds or thousands of kilometres with large regions of active convection adjacent to large regions where convection is rare or even absent altogether.

One particular feature of the tropical atmosphere is the so-called ‘Madden-Julian oscillation’ (MJO). This is not a regular oscillation, but a quasi-random variation in convection and in dynamical quantities such as wind and temperature, in which a region of active convection appears over the tropical Indian Ocean, drifts eastward into the western Pacific and then diminishes in strength over the eastern Pacific. The time between successive appearances of the active convection is typically 30-60 days. Most of the global circulation models used for climate prediction give very poor simulations of the MJO, suggesting that they poorly represent the physical processes that are responsible for it, probably because it depends on quite subtle interactions between convecting and non-convecting regions and between large scales and the scales of the weather systems within which active convection is embedded.

Given the complication of the tropical atmosphere – the range of spatial and temporal scales and the importance of cloud-scale processes including interactions between clouds and radiation – it
might seem that simple mathematical models would have limited relevance. However, provided the need for crude but simple representations of cloud-scale processes is accepted, relatively simple models can capture some of the important interactions between these processes and the large-scale dynamics and provide genuine insight into ways in which the representation of tropical circulations in global climate models might be improved.

One particular class of models that has been studied over the last 15 years or so, and is now being argued to provide a basis for understanding the MJO, are models in which include the simple fluid dynamical equations (e.g. as represented by the ‘shallow-water equations’) together with a moisture field that is transported with the fluid and affects the fluid dynamics by determining the heating. An interesting limit is when the fluid dynamics is treated as quasi-steady and the entire time evolution is controlled by the moisture field. In this limit simple wave motion is sometimes possible and these waves are described as ‘moisture modes’. There is now quite a large literature on ‘moisture modes’ and their behaviour according to different dynamical formulations (e.g. incorporating different physical processes) and there are also several papers which discuss possible moisture-mode models for the MJO.

An essay on this topic should start by surveying some of the basic papers that have studied moisture modes in different forms, trying to present a unified summary of the important features of the behaviour and how it depends on the physical ingredients incorporated in the model. The essay might then move on to discuss in more detail the extent to which moisture modes provide an explanation for the MJO and how these simple models might be used to advance understanding of the MJO and to improve its representation in climate models. (But a student writing this essay might choose to focus on other topics, such as the way in which convection-scale processes are represented in the simple models, or the extent to which moisture modes, or simple models that allow moisture modes along other sorts of behaviour, are useful to understand other aspects of the tropical atmosphere.)

Relevant papers are listed below. The introduction to [1] provides a short overview of work on moisture modes and cites several relevant papers. [2] is an early paper that considers a relevant simple model and identifies moisture-mode behaviour. [3] is a relatively mathematical paper that may be easier for a Part III Mathematics student to read than a paper written for atmospheric scientists. [4] and [5] are papers that propose moisture-mode models for the MJO. [6] is a paper that argues on the other hand that the physical processes incorporated into the models described in [4] and [5] may not be relevant to the MJO in the real atmosphere and offers an alternative.

Relevant Courses

Essential: An undergraduate course in fluid dynamics

Useful: Hydrodynamic Stability, Fluid Dynamics of Climate (Neither is absolutely essential, but a any student who is considering choosing this essay and who is NOT intending to take Fluid Dynamics of Climate is advised – and welcome – to discuss with the setter.)

References


Viscoplastic fluids are a branch of non-Newtonian fluids characterised by a ‘yield stress’, below which they behave as a rigid solid, and above which they flow like a viscous fluid. A great array of materials, from mud and industrial waste to toothpaste and whipped cream, exhibit some degree of viscoplasticity, and they can behave in quite a different manner to their Newtonian counterparts. The interaction of buoyancy forces with viscoplasticity leads to a fascinating array of dynamics and flow patterns and plays an important role in a wide range of problems, from magmatic flows and mud volcanism to the cooking of porridge on a hot plate.

This essay will begin with a review of the classical models for viscoplastic fluids, including shear-thinning ‘regularisations’ and the simple Bingham model. The essay should then discuss the linear-stability problem, with a thorough review and discussion of how the introduction of a yield stress stabilizes the problem to linear perturbations, but not to non-linear ones. After reviewing this core material, the candidate should explore extensions, including at least some of: the dynamics of different canonical convective systems (e.g. Rayleigh–Bénard, Rayleigh–Taylor, heated side walls); analytically tractable convective solutions; numerical advances and challenges in modelling more complex flows; experimental techniques and observations; and physical applications and implications.

Relevant Courses

**Essential:** None

**Useful:** Slow Viscous Flow, Fluid Mechanics of the Solid Earth, Perturbation Methods, Hydrodynamic Stability

References


Roughly 30% of the heat lost from the oceanic crust is removed in mid-ocean-ridge hydrothermal systems, which consist of convectively circulating oceanic fluid below the sea bed and above a magmatic heat source. Hydrothermal circulation at mid-ocean ridges affects the composition of the crust and the ocean, and contains a wealth of complex fluid mechanics.

This essay will explore the complex hydrodynamics of mid-ocean-ridge (MOR) systems. Given the complexity and number of different processes involved, the essay is fairly open-ended, but should certainly include as core material a review of the main processes involved in MOR hydrothermal systems and a discussion of the mathematics of convection in porous media, which provides the key mechanism of heat transfer. In light of the presence of both heat and salt in the system, the concept and basic mathematics of double-diffusive convection should also be discussed. Beyond this, there are a number of directions that candidates could develop this essay, for example by exploring: further details and more advanced analysis of convection and double-diffusion in porous media; models of the mechanics of heat transport and crystallisation in the magma chamber; details of compaction in the upper mantle and magmatic system (‘viscous compaction’ models); or the induced flow in ‘black smoker’ plumes above the sea floor.

**Relevant Courses**

*Essential:* Fluid Mechanics of the Solid Earth

*Useful:* Slow Viscous Flow, Fluid Mechanics of the Environment, Hydrodynamic Stability

**References**


70. **Deterministic Descriptions for Large-scale Behaviour of Stochastic Models** .................................................................

Dr R. Jack

Small particles (size < 1µm) in water move by Brownian motion. They follow apparently random paths, that can be described by stochastic differential equations or Langevin equations.
However, if one considers many such particles, their density $\rho$ follows a deterministic diffusion equation $(\partial \rho / \partial t) = D \nabla^2 \rho$. How can it be that particles move randomly but the density follows a deterministic equation?

At a field-theoretic level, this behaviour can be understood within a saddle-point approximation. For a more precise statement, one should realise that there is a finite probability that the particle density does not follow the diffusion equation, but this probability tends to zero as the number of particles $N \to \infty$. The nature of this limit is described by the probabilistic theory of large deviations, which quantifies the probability of very rare events, as reviewed in [1]. The application of this theory to systems of many particles is the content of macroscopic fluctuation theory [2].

This essay will discuss the extent to which macroscopic fluctuation theory resolves our original puzzle – how can large-scale behaviour be described deterministically, when the microscopic equations of motion are stochastic? Two related directions are the use of stochastic partial differential equations (with small noise) to describe the behaviour of the density as a function of time [3], and the rigorous mathematical theory of hydrodynamic limits [4].

**Relevant Courses**

*Essential:* Theoretical Physics of Soft Condensed Matter.

*Useful:* Stochastic Calculus and Applications.

**References**


**71. Dynamical Phase Transitions into Absorbing States .......................**

Dr R. Jack

The theory of equilibrium phase transitions is based on probabilities of configurations of a system, such as an Ising model. In dynamical phase transitions, one concentrates instead on the dynamical trajectories (or paths) that a system follows, as a function of time. This opens up a range of new possibilities.

Some famous experiments in 2005 revealed a surprising new phase transition [1] in a system of small particles (size $\sim 0.2$mm). In one of the phases ("active phase") the particles move around, apparently at random. In the other phase ("inactive"), the particles move along periodic trajectories; they never collide with each other, so they keep returning to the same places. These experiments were understood in terms of a theoretical model [2] in which a similar transition takes place. It is now understood that this transition is accompanied by a diverging correlation length (and an associated correlation time), similar to equilibrium phase transitions. However, the theory of equilibrium phase transitions does not apply here.
There are field theories that account for the dynamical trajectories of the system and can de-
scribe critical points of this type, but there are significant open questions about the universality
class that is relevant for these models and experiments [3]. Recent studies have also uncovered
new properties of the critical point, such as “hyperuniformity” of the inactive phase [4].

This essay will review the theory of these dynamical phase transitions, the phenomena that
occur near the critical point, and the connection with experimental and computer simulation
data. The theoretical treatments are based on field theories or stochastic partial differential
equations – understanding which terms are “relevant” in these theories can be much more
tricky than the standard approaches that work for equilibrium phase transitions.

**Relevant Courses**

*Essential:* Theoretical Physics of Soft Condensed Matter

*Useful:* Statistical Field Theory

**References**


020601 (2017).

72. Renormalization Group Study of Constrained Hamiltonians ..............

Dr R. Jack and Dr E. Tjhung

At criticality, or second order phase transition point, the correlation length of a physical system
diverges as power law $\xi \sim |T - T_c|^{-\nu}$ ($T_c$ being the critical temperature). $\nu > 0$ is a critical
exponent whose value is independent of the microscopic details of the system. In other words, at
criticality, many different physical systems share the same critical exponents (universality class).
It has been thought for a long time that different universality classes can be determined solely
from their symmetries [1]. For instance, liquid/gas critical point and Ising model at critical
temperature correspond to up/down symmetry and para-ferromagnetic transition corresponds
to rotational symmetry. However it was recently discovered that symmetry alone may not
uniquely define the universality class of the system. For instance, one can imagine a Hamiltonian
with the same rotational symmetry as in a para-ferromagnetic transition but with an additional
constraint $\nabla \times \mathbf{m} = 0$ [2]. This constraint gives a new fixed point in the renormalization group
flow, which indicates a different universality class from the unconstrained Hamiltonian (even
though they have the same symmetry). Topologically, the constraint $\nabla \times \mathbf{m} = 0$ suppresses any
vortices/spiral defects which may be created in the system.

Similarly, one can also imagine the same Hamiltonian with another constraint $\nabla \cdot \mathbf{m} = 0$ [3] and
this will give yet another different fixed point in the renormalization group flow. Topologically
speaking, this constraint will kill off any hedgehog defects in the system. In this essay, we will
investigate how these two different pictures will fit together.
Bell’s theorem shows that locally causal hidden variable theories cannot reproduce all the predictions of quantum theory for measurements on separated entangled states, for example the singlet state of two spin $\frac{1}{2}$ particles. Bell inequalities give bounds on correlations obtainable from locally causal hidden variable theories for specific classes of measurements, which are violated by some quantum states.

Recently, Kent and Pitalua-García considered measurements of pairs of separated spin $\frac{1}{2}$ particles about two spin axes $\mathbf{a}, \mathbf{b}$, chosen randomly subject to the constraint that $\mathbf{a} \cdot \mathbf{b} = \cos(\theta)$. They showed that Bell inequalities can be proven for such measurements, for a wide range of $\theta$. However, the strongest possible bounds are not known for most values of $\theta$.

There is a natural geometric and pictorial interpretation of this question in terms of colourings of the Bloch sphere and “jumps” through an angle $\theta$, in which one can picture the jump as if made by a grasshopper that landed randomly on the sphere and jumps through fixed angle $\theta$ in a random direction. This interpretation is intriguing enough in its own right to motivate considering the problem outside the original context of quantum theory and Bell inequalities.

For example, Goulko and Kent considered the “grasshopper problem” on the plane, and were able to prove some analytic results and obtain extensive numerical evidence about the form of apparently (near-)optimal colourings.

An ideal essay would review the literature to date and produce interesting extensions of the known results, most likely through new numerical solutions.
74. Does the Miles-Howard Theorem Have any Relevance to Turbulent Stratified Shear Flows? .......................... 
Professor R. R. Kerswell

The Miles-Howard theorem (e.g. Howard 1961) is a classical result in stably-stratified flows which gives sufficient conditions for linear stability of a steady, inviscid, unidirectional stratified shear flow. If the local Richardson number ($R_i$) of the flow (a non-dimensional parameter measuring the relative strength of the stable stratification to the shear) is everywhere larger than $\frac{1}{4}$, the Miles-Howard theorem asserts that the flow has to be linearly stable. While this result has been derived in very idealised circumstances (e.g. the absence of fluid viscosity), the prediction of stability for $R_i > \frac{1}{4}$ seems to hold some greater truth. Many observations seem to show turbulent flows operating at or just below $R_i = \frac{1}{4}$ suggesting that this value may actually approximate some sort of nonlinear stability barrier and that the turbulence is marginally stable. This part III essay would explore the literature (starting with the references below) to discuss the evidence for these statements.

Relevant Courses

Essential: Fluids II, Hydrodynamic Stability, Methods
Useful: Asymptotics, Dynamical Systems

References


75. Self-Sustaining Processes in 2 Dimensional Shear Flows ................. 
Professor R. R. Kerswell

Understanding how and when simple shear flows break down to turbulence is an ongoing problem of huge importance in understanding the environment and industrial applications. Much of the recent progress made in understanding transition to turbulence in linearly-stable shear flows has followed from uncovering a ‘self-sustaining process’ which stops solutions unconnected to the basic shear flow from decaying away through viscous damping. This process, first found by Waleffe (1997) and later recognised as the vortex-wave-interaction theory of Hall and Smith, is inherently 3-dimensional. In 2 dimensions, transition is usually only found when there is a linear instability of the basic shear flow (e.g. channel flow). However, there are situations where non-decaying solutions unconnected to a linearly-stable shear flow exist indicating a different type of...
Deguchi & Walton (2013) describe just such a situation - axisymmetric annular sliding Couette flow - and identify these non-decaying states via tracking bifurcations from other parts of parameter space. They also describe the asymptotic structure of the non-decaying solutions. The aim of this part III essay would be to try to deconstruct this 2D self-sustaining process into its component parts with the aim of generating the equivalent cartoon to Waleffe’s (1997) figure 1 for his 3D process (streamwise rolls generate streaks which become unstable to waves which then nonlinearly self-interact to drive the original rolls). This essay is a great opportunity for somebody interested in mathematical fluid dynamics and asymptotics to immerse themselves in a very current topic.

**Relevant Courses**

*Essential:* Asymptotics, Fluids II, Hydrodynamic Stability, Methods

*Useful:* Dynamical Systems

**References**


**76. Thermocapillary Instabilities** .................................................................

Dr K. Kowal

An interface between two immiscible fluids is subject to interfacial, or surface, tension, which may depend on various scalar fields, such as the temperature and solute concentration, as well as on the concentration of surfactants, or compounds that decrease surface tension. When surface tension depends on the temperature, gradients along the interface induce shear stresses that result in thermocapillary fluid flow. Thermocapillary flows are ubiquitous in nature and industry, such as in crystal growth, welding, the manufacture of silicon wafers, electron beam melting of metals and the rupture of thin films. In many of these, the transport of heat can increase significantly as a result of additional mixing processes triggered by thermocapillary instabilities. The onset of these instabilities can be examined using linear stability theory. The essay should review the mechanisms of the instability using linear stability theory as well as characterise the different modes of instability that occur in *static* and *dynamic* thermocapillary liquid layers. The candidate may choose to examine the long-wave evolution of thermocapillary waves, their nonlinear dynamics, or the effects of surfactants and thermophoresis (the Soret effect). The exact direction of the essay depends on the interests of the candidate.

**Relevant Courses**

*Essential:* Undergraduate Fluid Mechanics

*Useful:* Hydrodynamic Stability, Slow Viscous Flow
References


77. Capillary Retraction of Viscous Fluid Sheets .........................

Professor J. R. Lister

When a hole forms in a soap film, a capillary force/length $2\gamma$ acts on the edge of the hole, pulling it outward and rapidly increasing the size of the hole. Early work by Taylor, Culick and then Keller looked at the inviscid dynamics appropriate to watery soap films. More recently, there has been considerable interest in viscous dynamics, whether for rupture of very viscous films, retraction of the edge of a sheet of molten glass, or expansion of the ‘hole’ in the sheet of external fluid between coalescing bubbles or drops.

The essay should review the development of theoretical modelling for the viscous case (i.e. large, though possibly finite, Ohnesorge number) and discuss the range of applications. It should include a derivation for the case of radial sheet flow analogous to that provided for axisymmetric extensional flow in lectures. Some angles to explore in the literature might include the differences between two-dimensional and radial flow, the effect of the initial sheet thickness profile, the boundary condition at the edge, and the role of inertia when the Ohnesorge number is large but finite. The essay should conclude with a discussion of the extent to which the theory explains experimental observations, and where the theory needs to be developed further.

Relevant courses

*Essential*: Slow Viscous Flow

References


78. The Shape of a (Viscous) Chocolate Fountain

Professor J. R. Lister

In a chocolate fountain, molten chocolate flows down over a vertical stack of dome-shaped tiers of increasing size (Try googling images for ‘chocolate fountain’). The flow on each tier can be described very simply using lubrication theory. The chocolate flows over the circular edge of each tier to form a thin axisymmetric curtain of falling fluid, which falls until it lands on the next tier below. Observation shows that the curtain contracts inwards as it falls, with radius looking almost linear as a function of height. The theory for an inviscid curtain of fluid is well-established from the study of ‘water bells’, but molten chocolate is viscous!

This essay would likely take the form of a mini-project to calculate the shape of a falling very viscous Newtonian fluid curtain by using the equations of viscous extensional flow, as adapted to the axisymmetric geometry. The equations of axisymmetric shell theory in elasticity would be a useful comparison. Inertia should be neglected, but surface tension is (probably) relevant. The problem differs from tube-drawing in that radial and axial variations are comparable. Further references and guidance are available on request.

Relevant courses

Essential: Slow Viscous Flow

References

Audoly, B. & Pomeau, Y. 2010 Elasticity and geometry OUP

79. Kinks with Long-Range Tails

Professor N. S. Manton

Kinks, the basic type of topological soliton in one space dimension, are classical solutions of a scalar field theory with multiple vacua. Kinks usually have exponentially localised, short-range tails, but recently there has been an effort to better understand kinks having long-range tails with power-law fall-off. Such tails arise when the field is massless to the left or right of the kink. The interaction of a kink with another kink, or with an antikink, is tricky to calculate, because the long-range tail obeys a nonlinear equation. In this essay you should review how kinks with long-range tails arise, and the attempts to understand their interactions through numerical and
analytical studies. A comparison with the easier case of kinks having short-range tails will be worthwhile.

**Relevant Courses**

*Essential:* None

*Useful:* Quantum Field Theory; Classical and Quantum Solitons

**References**

Kinks are described in most field theory books discussing solitons, e.g.


and the books by R. Rajaraman, T. Vachaspati, E. Weinberg etc. Kinks with long-range tails were considered by


and for recent discussions, see


80. **Wave Attractors in Rotating and Stratified Fluids**

Professor G. I. Ogilvie

Internal waves can propagate in rotating and/or stably stratified fluids (as often occur in astrophysical and geophysical settings) as a result of Coriolis and/or buoyancy forces. Their properties are radically different from those of acoustic or electromagnetic waves. The frequency of an internal wave depends on the direction of the wavevector but not on its magnitude. Waves of a given frequency follow characteristic paths through the fluid and reflect from its boundaries. In many cases the rays typically converge towards limit cycles known as wave attractors. One application of this finding is to tidally forced fluids in astrophysical and geophysical settings. If tidal disturbances are focused towards a wave attractor, this can lead to efficient tidal dissipation that in some cases is independent of the small-scale diffusive processes.

This essay should review the subject of internal wave attractors, including some of the more recent developments. Some simple explicit examples should be provided, which could involve original calculations. Topics that might be covered include:

1. The behaviour of rays for pure inertial waves in a uniformly rotating spherical shell.

2. The relation, if any, between the propagation of rays within a container and the existence of invisicid normal modes.
3. The consequences of a wave attractor for the decay rate of a free oscillation mode, or the dissipation rate of a forced disturbance, in the presence of a small viscosity.

4. The roles of nonlinearity and instability in wave attractors.

5. The relevance of wave attractors to tidal dissipation in astrophysical systems.

**Relevant Courses**

*Essential:* None

*Useful:* Astrophysical Fluid Dynamics

**References**


81. **Eccentric Astrophysical Discs** ..................................................  

*Professor G. I. Ogilvie*

Closed Keplerian orbits around a massive body are generally non-circular. A thin Keplerian disc may be composed of nested elliptical orbits whose eccentricity $e$ and longitude of periapsis $\varpi$ vary continuously with semi-major axis $a$ and time $t$. The complex eccentricity is $e \exp(i\varpi) = E(a,t)$.

When $|E|$ and $|\partial E/\partial \ln a|$ are sufficiently small, a linear evolutionary equation can be derived for the complex eccentricity, which determines how the shape of the disc propagates by means of pressure, viscosity, self-gravity and other collective effects that are weak compared to gravity. More generally, $E(a,t)$ satisfies a nonlinear evolutionary equation and the presence of eccentricity affects the transport of mass and angular momentum in the disc.

Eccentric discs are thought to exist in many astrophysical situations, including narrow planetary rings around Saturn and Uranus, protoplanetary discs around young stars, circumstellar discs around rapidly rotating Be stars, accretion discs around compact objects in close binary systems, circumbinary discs around binary black holes or young binary stars, and debris discs around white dwarfs.

This essay should discuss aspects of the dynamics and significance of eccentric discs in at least one of these areas of application. Apart from the derivation and interpretation of the evolutionary equation(s) for eccentric discs, theoretical topics that might be discussed include the stability of fluid flows with elliptical streamlines, the gravitational interaction of orbiting companions with a disc, and the numerical simulation of eccentric discs.

A selection of recent references is provided below; use of the NASA ADS archive to locate other relevant publications is recommended.
Relevant Courses

Essential: None
Useful: Astrophysical Fluid Dynamics, Dynamics of Astrophysical Discs, Planetary System Dynamics

References


82. The BMS Group, the Soft Graviton and Gravitational Memory

Professor M. J. Perry

The Bondi-Metzner-Sachs group is the group of asymptotic symmetries of a spacetime that is asymptotically flat. It should be thought of as a generalization of the Poincaré group of symmetries of Minkowski spacetime. A gravitational wave escaping to null infinity will generate a BMS transformation that can be interpreted as a form of gravitational memory. Curiously, the BMS group is also related to the notorious infrared divergences encountered in the quantum field theory of gravitation. Weinberg’s soft graviton theorem explains how these divergences can be cancelled. The essay should explore and develop these concepts and show how they can regarded as different faces of the same fundamental idea.

Relevant Courses

Essential: General Relativity, Quantum Field Theory
Useful: Advanced Quantum Field Theory

References


83. Defining the Anderson-Bergmann Velocity Poisson Bracket

J. B. Pitts

Do we yet know what a Poisson bracket ought to mean? The oft-cited 1951 Anderson-Bergmann paper on constrained Hamiltonian dynamics contains a little-noticed ‘Poisson bracket’ for velocities, \( \{ \dot{q}, F \} = \frac{d}{dt} \{ q, F \} \). This bracket appears to yield only correct answers, to be indispensable
(apart from a near-equivalent involving secret introduction of temporal smearing functions) for answering some important questions (such as the gauge in/covariance of Hamilton’s equations and the canonical Lagrangian $\dot{p}q - H$), to violate the usual Poisson bracket product rule occasionally in order to satisfy the time differentiation product rule, and to have no clear mathematical basis.

Discuss these issues.

**Relevant Courses**

*Essential:* None

*Useful:* Hamiltonian General Relativity

**References**

Besides references from the useful course, see:


84. **Validation and Extensions of the Parabolic Equation Method in Random Media ..........................................................**

Dr O. Rath Spivack

Problems of wave propagation in inhomogeneous media are difficult to solve analytically. Solutions based on the parabolic wave equation (PWE) method, have been used successfully in many cases, including the propagation of acoustic waves in the ocean, and acoustic or optical waves in turbulent atmosphere.

In the absence of exact solutions, a major challenge is the validation of different approximations, and evaluating their accuracy. Since the early work of Tappert [1], several developments have improved the efficiency of numerical implementation, for example through the introduction of split-step methods. Other developments have extended the scope of the approximation to include wide angle geometries and backscatter or reverberation, for example through higher order approximations and two-way parabolic equations. Further research has been devoted to dealing with the multiscale nature of the wave propagation problem when the wavelength is much smaller than the range over which signals are measured, and the scale of the inhomogeneities is comparable to the wavelength.

This essay should focus on propagation in random media. After an introduction to the PWE method and an overview of approximation methods, it should choose specific examples for which to explain ways of estimating accuracy and compare with other available approximation.

A few possible references are given below, but more specific references depending on the choice of focus will be provided.
Relevant Courses

Essential: Knowledge of the wave equation and basic concepts in wave propagation, from any course.

Useful: The Part III course Direct and Inverse Scattering of Waves

References


Dr O. Rath Spivack

Many inverse problems in mathematical physics can be formally expressed as

$$Ax = y,$$  \hspace{1cm}(1)

where $A$ is an operator from a normed vector space $X$ into a normed vector space $Y$, $y \in Y$ is given data, often measured data, and $x \in X$ is the unknown. Usually such problems are ill-posed, and various methods are used to ensure existence and uniqueness, as well as stability of the solution, which is referred to as ‘regularisation’.

Regularisation can in some cases be achieved by projection onto finite-dimensional subspaces $A_n \subset X$. Krylov subspace methods are iterative methods in which the solution is sought by successive approximations $x_n \in K_n$, where $K_n$ is the Krylov subspace $span\{d, Bd, \cdots B^{n-1}d\}$, with $B$ and $d$ dependent on $A$ and $y$. The Conjugate Gradient method and its variants are examples of Krylov subspace methods, and other have also been used, sometimes together with Tikhonov regularisation.

This essay should explore the regularising properties of Krylov subspace methods, focusing on some particular issues according to personal interests and background. This could also be, for example, applications of Krylov methods to practical inverse problems.

A few example references are given below, and more will be provided depending on the choice of focus.

Relevant Courses

Essential: Basic knowledge of linear analysis, from any course.

Useful: The Part III courses Inverse Problems in Imaging, Direct and Inverse Scattering of Waves, Mathematics of Image Reconstruction (non-examinable)
86. Dimensional Reduction

String and Superstring theories exist in higher dimensional spacetimes. This requirement is forced upon us by the quantum consistency of these theories. One way to make contact with the four-dimensional physics of experience is to assume that these extra dimensions take the form of small compact geometries that are thus far beyond our ability to detect directly. The study of how lower dimensional effective theories depend on the details of the compact space and how one might realise known theories as compactifications of higher dimensional theories is the subject of dimensional reduction (often referred to as Kaluza-Klein theory). One of the most intriguing aspects is the realisation of non-gravitational physics, such as Yang-Mills theory, as purely gravitational physics in higher dimensions.

The utility of dimensional reduction extends far beyond the requirements of string theory. Dimensional reduction gives a way of understanding the existence of certain supergravity theories in terms of the dimensional reduction of a higher dimensional theory and gives a concrete way of constructing new supergravity theories from known ones in higher dimensions. More recently, the desire to understand certain supergravity theories as arising as compactifications of string theory have suggested the existence of string theory backgrounds which cannot be understood in terms of a worldsheet embedding into a conventional Riemannian manifold, pointing towards an intrinsically string-theoretic generalisation of Riemannian geometry.

The first part of the essay will include a study of the dimensional reduction of simple gravitational theories on tori. The essay may then proceed in many different directions. One possible direction is to study compactifications on manifolds with non-abelian isometries and the relationship with gauge theory and gauged supergravities in the lower dimensions. Simple twisted or Scherk-Schwarz compactifications might also be studied. The question of consistency, in the sense that solutions to the lower dimensional equations of motion lift to solutions of the higher dimensional theory could be discussed. The interested essayist might choose to investigate the relationship between compactification on Tori and the duality symmetries of string theory.

Relevant Courses

Essential: Part III General Relativity, Part III Quantum Field Theory
Useful: Part III String Theory, Part III Advanced Quantum Field Theory
87. Strong Cosmic Censorship in Asymptotically de Sitter Spacetimes

Dr J. E. Santos

All observed gravitational dynamics in our Universe appears consistent with Einstein’s theory of General Relativity, possibly endowed with a positive cosmological constant. However, the appearance of Cauchy horizons in certain solutions of the Einstein equation signals a potential breakdown of determinism within GR - the future history of any observer that crosses such an horizon cannot be determined using the Einstein field equation and the initial data! Penroser’s Strong Cosmic Censorship conjecture proposes that solutions containing such horizons cannot be dynamically generated starting with generic initial data.

The essay should do a thorough review of the recent proposed scenarios for violating the Strong Cosmic Censorship conjecture, and should be written in a language accessible to other Part III students taking similar courses.

Relevant Courses

Essential: General Relativity and Black Holes

References


88. Confinement

Dr D. B. Skinner

The microscopic degrees of freedom of QCD are quarks and gluons. In the world around us such particles are never seen individually, but are always trapped within composite hadrons such as protons, neutrons and pions. This phenomenon is not seen in perturbation theory, and a complete understanding remains one of the outstanding challenges of theoretical physics. This essay will explore aspects of confinement in various contexts, from solvable toy models such as the $\mathbb{CP}^n$ model in $d = 2$, Polyakov’s confinement mechanism in the Abelian Higgs model in $d = 3$, to ’t Hooft’s picture of the $d = 4$ QCD vacuum as a dual superconductor.
89. Computational Complexity of Quantum Ball Permuting Model

Dr S. Strelchuk

The study of the computational power of many physical systems including noninteracting fermions, noninteracting bosons, or anyons in a 2+1 dimensional quantum field theory [1-4] can be recast as a study of models of computation based on permuting distinguishable particles – much like quantum ‘balls’ in boxes [5]. Some of these systems turn out to be solvable or integrable, and thus regarded as simple from the mathematical physics point of view, but they are nevertheless interesting from the complexity-theoretic point of view: their computational power appears to lie between classical and quantum computing.

The essay should introduce the quantum ball permuting model [5] and explain how it models actual physical processes. It should then discuss the computational complexity of one or two physical systems [1-4] that can be described in this framework.

Relevant Courses

Recommended: Part III Quantum Computing

References

Modern quantum algorithms require computational resources which are currently beyond the reach of state-of-the-art implementations. But even minimal quantum resources can be made useful if we use them in conjunction with powerful classical optimization methods. This approach has been exploited in Variational Quantum Eigensolver (VQE) [1]. It makes use of Ritzs variational principle to prepare approximations to the ground state and its energy. In this algorithm, the quantum computer is used to prepare a class of variational ‘trial’ states which are characterized by a set of parameters. Then, the expectation value of the energy is estimated and used by a classical optimizer to generate a new set of improved parameters which are then used to prepare the next iteration of trial states. The advantage of VQE over purely classical simulation techniques is that it is able to prepare trial states that cannot be generated by efficient classical algorithms.

This essay should discuss the algorithm and its applications [2-3].

**Relevant Courses**

*Recommended:* Part III Quantum Computing

**References**


**91. Lattice QCD and Hadron Spectroscopy**

*Dr C. E. Thomas*

Quantum chromodynamics (QCD) is a quantum field theory that exhibits many interesting phenomena such as asymptotic freedom and confinement. Moreover, it describes the strong interaction of particle physics, i.e., how quarks and gluons interact and give rise to the non-trivial structure and dynamics of hadrons. Recent observations of a number of ‘exotic’ structures have generated a lot of interest and hadrons are currently the subject of many theoretical and experimental investigations.

Computing the masses and other properties of hadrons within QCD is a long-standing challenge because the QCD coupling is strong in the relevant low-energy regime. Lattice QCD is a non-perturbative technique that enables first-principles computations of the properties of hadrons.
using Monte Carlo methods. This essay should provide a brief introduction to lattice QCD and then discuss its application to calculating the spectra of hadrons.

### Relevant Courses

**Essential:** Quantum Field Theory

**Useful:** Advanced Quantum Field Theory; Symmetries, Fields and Particles; Standard Model

### References


[7] Useful papers, including reviews, can be found online on the arXiv (http://arxiv.org/).

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**92. Chiral Fermions on the Lattice**  

**Professor D. Tong**

No one knows how to write down a discrete version of the laws of physics. The problem is that the Standard Model is a “chiral gauge theory”, meaning that the left-handed and right-handed fermions experience different forces. There are topological reasons, enshrined in the Nielsen-Ninomiya theorem, which mean that it is difficult to construct discrete (or “lattice”) versions of such theories. The purpose of this essay is to explain this problem and to describe attempts to circumvent it.

### Relevant Courses

**Essential:** Quantum Field Theory, Advanced Quantum Field Theory, Standard Model

### References

[1] An introduction to lattice fermions can be found in chapter 4 of the lectures: http://www.damtp.cam.ac.uk/user/tong/gaugetheory.html

[2] Lectures on chiral lattice fermions, focussing on the domain wall and overlap approaches, can be found in:

M. Lüscher, "Chiral Gauge Theories Revisited", hep-th/0102028

D. Kaplan, "Chiral Symmetry and Lattice Fermions", arXiv:0912.2560
93. Strichartz Estimates and Nonlinear Schrödinger Equations

Dr C.M. Warnick

Nonlinear Schrödinger equations arise as models for various physical phenomena including Bose-Einstein condensates, light propagation in optical fibres, waves on deep inviscid water and Langmuir waves in hot plasmas.

Strichartz estimates are estimates for the linear Schrödinger equation which control \( L^p_t L^q_x \) norms in terms of the data. These estimates capture the dispersive behaviour of solutions to the linear Schrödinger equation, and are of considerable importance in the study of the nonlinear problem. The first such estimate was established in [1], but since then many refinements have been established, see for example [2, 3] and references therein.

The purpose of this essay is to investigate Strichartz estimates for the Schrödinger operator and applications to semilinear Schrödinger equations. As a typical application, the local well-posedness [2] may be considered, but a more ambitious option would be to look at results concerning global well posedness and scattering for the power law nonlinearity in the subcritical [4] or critical case [5].

Relevant Courses

Essential: Analysis of PDE
Useful: Distribution Theory and Applications

References


94. Linear Fields on Black Hole Backgrounds

Dr C.M. Warnick

The first step to understanding black hole stability is to study in detail the behaviour of linear fields on a fixed black hole background. In recent years, our understanding of this problem has advanced substantially. Through the vector field method and its refinements, considerable progress has been made to understand solutions of the main model equations: the wave equation, Maxwell’s equation and the linearised gravitational equations. A robust understanding has been obtained of the decay properties of fields in the black hole exterior for a variety of black hole
solutions to Einstein’s equations, with and without cosmological constant. There has also been significant work to understand the behaviour of linear fields in the black hole interior, with implications for the strong cosmic censorship conjecture.

A good essay will survey a selection of recent results, and treat at least one result in detail.

**Relevant Courses**

*Essential:* General Relativity, Black Holes

*Useful:* Analysis of PDE

**References**


95. **Stationary Spacetimes and Finsler Geometry**

*Dr C. M. Warnick*

For any spacetime, it is important to try and understand the behaviour of light rays. In a static spacetime, the problem of finding the null geodesics can be reduced by a conformal transformation to studying the geodesics of a related Riemannian metric, which defines the optical geometry of the spacetime. For a stationary spacetime, the optical geometry is not Riemannian, but rather defined by a type of Finsler metric.

Finsler geometry generalises Riemannian geometry by not requiring the line element to be defined by a quadratic form. It arises naturally in certain variational problems. A good attempt should include the basic definitions and results from Finsler geometry, the connection to stationary spacetimes as well as some further applications.

**Relevant Courses**

*Essential:* General Relativity

*Useful:* Black Holes, Differential Geometry

**References**

96. Markov Chain Monte Carlo for Tall Data .......................... Dr S. A. Bacallado

Markov chain Monte Carlo methods can be credited with the popularity of Bayesian inference since their development in the 1990s, but they remain computationally expensive in many applications. Standard algorithms do not scale well with tall data, that is, when there are large numbers of conditionally independent observations, as they require computing the full likelihood at each iteration.

This essay is meant to review a set of scalable algorithms for Bayesian inference developed in the last 4 years, starting from the review of Bardenet et al. [1], and focusing on a few of the methods discussed in [2-6] or other references in the review. Those so inclined can implement a subset of the methods and present a numerical experiment or practical application.

Relevant Courses

*Essential:* Bayesian Modelling and Computation

References


97. Chaining and Metric Entropy .............................. Professor R. Nickl

Many mathematical challenges in contemporary high-dimensional and non-parametric statistics involve the control of the stochastic size of suprema of collections of random variables, that is,
of expressions of the form $E \sup_{t \in T} X_t$, where $(X_t : t \in T)$ is a stochastic process indexed by the same general index set $T$. For centred Gaussian processes this question can be answered entirely in terms of the complexity of the metric space $(T, d)$, where $d^2(s, t) = E(X_s - X_t)^2, s, t \in T$. For other stochastic processes, such as empirical or Rademacher processes, characterising the size of such suprema can be a more delicate problem. For sub-Gaussian processes a universal tool that often provides statistically useful bounds is Dudley’s metric entropy bound based on chaining and on Kolmogorov’s notion of the $d$-metric entropy of the set $T$. More sophisticated techniques, such as ‘generic chaining’, are sometimes required to sharpen such results.

The purpose of this essay is to summarise some main ideas in this area in a coherent way, and to try to explain some of the proofs, as well as potential applications to high-dimensional and non-parametric statistics. References that include the key material are [3, 4] and applications are presented in [2,4,5]. A more ambitious student can also look into some exciting recent work on the problem [1,6,7].

**Relevant Courses**

Some background in analysis, statistics and probability is helpful.

**References**


98. **Instantons** .................................................................

Professor N. Dorey

Instantons are finite action classical solutions of the equations of motion of a field theory in Euclidean spacetime [1,2]. At weak coupling, they appear as saddle points of the path integral and yield exponentially small contributions to observables. However, in the presence of fermions, they can contribute to quantities which are not corrected at any finite order in perturbation theory. This is particularly true in supersymmetric theories where they sometimes yield the exact results for special protected observables (see [3,4] for a review). Instanton solutions in four dimensional gauge theory are also of interest to mathematicians; thanks to the ADHM construction [5], a general solution is available on $\mathbb{R}^4$ and, on more general four-manifolds, the moduli space of instanton solutions is the starting point for constructing topological invariants
known as Donaldson invariants (see e.g. [6]). Instanton effects are also important in string theory where they play a key role in the phenomenon of mirror symmetry [7]. The essay should start with the basics of instanton physics, but can progress to survey one or more advanced topics.

**Relevant Courses**

*Essential:* Quantum Field Theory, Advanced Quantum Field Theory  
*Useful:* Solitons, Supersymmetry, String Theory

**References**


[3] David Tong, ”TASI lectures on Solitons”, Lecture 1  
[http://www.damtp.cam.ac.uk/user/tong/tasi/instanton.pdf](http://www.damtp.cam.ac.uk/user/tong/tasi/instanton.pdf)


### 99. Statistical Inference for Discretely Observed Compound Poisson Processes and Related Jump Processes

Dr A. J. Coca

Stochastic processes are used in innumerable applications to model random dynamical systems. In many of these applications random shocks take place and a class of particular importance is that of jump processes; prominent examples arise from seismology, neuroscience, finance, queues, telecommunication networks, radiation detection, and many more. The data acquisition systems of most of these only register values of the underlying jump process at discrete times and statistical inference for the process becomes a statistical (generally nonlinear) inverse problem. This is an active area of research and, in particular, nonparametric inference for discretely observed compound Poisson processes (CPPs) has received much attention since the seminal work in [1]. The importance of CPPs is two-fold: they are one of the “off-the-shelf” processes first used in practice to model jumps; and, from a more theoretical perspective, they sit at the intersection of fundamental classes of jump processes such as Lévy processes, renewal processes, Poisson point processes, etc. whilst they still retain much of the mathematical structure and challenges of each class constituting a tractable but representative model to work with.

I envisage considerable flexibility in this project: for a student wishing to acquire a general overview, they could review the literature on non-Bayesian estimation of CPPs (see [1,2,3,4,6]), potentially connecting it to respective literature on more general Lévy or renewal processes; for a more technical project, they could review some of the literature in more detail focusing on a concept from nonparametric statistics such as adaptive estimation (see [3,4]), uncertainty quantification (see [1,2]), Bayesian inference (see [7,8]) or information bounds (see [5,9]); and, those with a more computational taste could implement and compare existing methodologies.
including spectral ([2,6]), Bayesian ([7,8]), data-augmentation ([7]) and model selection techniques ([3,4]), and others less studied such as the bootstrap and optimisation methods. More references are available and there is room for some novel work in all these directions.

**Relevant Courses**

*Essential:* None

*Useful:* Topics in Statistical Theory and Advanced Probability.

**References**


100. On the structure of Chevalley groups over local fields .................

Dr B. L. Romano

The main goal of this essay is to read and understand the paper “On some Bruhat decomposition and the structure of Hecke rings of \( p \)-adic Chevalley groups” by Iwahori and Matsumoto. To do so, you’ll first learn about the structure of Chevalley groups (e.g. root groups, commutator relations, \((B,N)\)-pairs) by reading, e.g. Carter’s *Simple groups of Lie type* (Steinberg’s notes are also a good reference). You’ll also need a basic understanding of local fields: students who have not taken a course covering this topic can find details in, e.g., Serre’s *Local fields*. Your essay should discuss the concepts in Iwahori–Matsumoto (e.g. affine Weyl groups and root systems, maximal compact subgroups) via explicit examples. Interested students should also gain some context for the importance of Iwahori–Matsumoto’s results by reading about the representation theory of \( p \)-adic groups: Kim’s “Supercuspidal representations: construction and exhaustion” is a nice survey paper for this, and Chapter 1 of Bushnell–Henniart’s *The local Langlands conjecture for GL(2)* is good background material.
Relevant Courses

**Essential:** Lie algebras and their representations

**Useful:** Algebraic number theory (or any course that provides background in local fields)

References


101. Kodaira’s problem .........................................................

**Dr R. Svaldi**

Kähler manifolds are the main class of objects that are studied in complex geometry because of their rich structure. A large source of examples of Kähler manifolds is given by projective ones, i.e., those that can be embedded holomorphically in some projective space. It is natural to ask exactly how large the class of smooth projective varieties is within that of compact Kähler manifolds. Kodaira showed that there is a cohomological criterion to determine when a compact Kähler manifold is projective. But projectivity is not stable under deformations of the complex structure.

In the course of his analysis of compact Kähler surfaces, Kodaira, [2], proved that every such surface $S$ possesses deformations that become projective. This result was later generalized by Buchdahl, [1].

Kodaira’s problem asks whether this phenomenon occurs in (complex) dimension higher than 2. Voisin, [4], has constructed a series of examples in dimension 4 and higher for which Kodaira’s problem has a negative answer. She later proved that even the birational version of the problem – that is, if a birational model of a compact Kähler manifold can be deformed to a projective variety – has also negative answer, see [5].

In this essay, you will present Voisin’s constructions, after introducing Kodaira’s problem in the framework of Kähler geometry.

Relevant Courses

**Essential:** Part III Algebraic Geometry, Algebraic Topology, and Complex Manifolds.

**Useful:** Part III Algebra.
102. Semipositivity properties of the tangent bundle

Dr R. Svaldi

In the study of algebraic varieties, a huge role is played by the positivity properties of the tangent bundle of an algebraic variety. The word positivity here refers to the existence of metrics with positive curvature on some portion either of the tangent bundle or of its tensor/exterior powers. For example, if at the general point of a smooth projective variety there are curves along which the determinant of the tangent bundle has positive curvature then the variety itself is uniruled, that is, is covered by rational curves, see [5]. Actually, one of the central conjectures in the classification of algebraic varieties, classically attributed to Mumford, predicts that the same conclusion should hold under a much weaker condition: namely, when the determinant of the cotangent bundle and its tensor powers have no sections.

The aim of this essay is to explore some of the fundamental theorems and techniques that describe the properties of the tangent bundle in relation to its positivity, or lack thereof.

One of the first results in this area – that should constitute the core of the essay – is Miyaoka’s Semipositivity Theorem, [4, 5]. In order to give a proof of this theorem, you will have to familiarize yourself with a plethora of techniques and results that are at the foundations of modern birational geometry, see [1], for example.

This is a rather advanced essay, hence, ideally, you would already be comfortable using tools from algebraic geometry at the level of chapter 2-3 of Hartshorne and beyond.

Relevant Courses

Essential: Part III Algebraic Geometry, and Algebra.
Useful: Part III Complex Manifolds, and Algebraic Topology.

References

103. Introduction to the Minimal Model Programme

Dr R. Svaldi

In algebraic geometry, one of the main goals is to classify algebraic varieties. Among the many possible approaches to this problem, one can choose to consider two varieties to be equivalent when they are birational, that is, when the fields of rational functions are isomorphic. The idea is then that, within a given equivalence class of birational varieties, one should identify a representative whose geometric structure is simple to analyze and to break down into simpler pieces. Such models are conjectured to exist and their construction is the aim of the so-called Minimal Model Programme (MMP in short), initiated by Mori in the late 1970’s and then developed by many authors in the course of the past 40 years.

In this essay, you will look at some of the ideas and the techniques behind the MMP.

You will first introduce the notion of log pair and of singularities of pairs. This is a fundamental tool nowadays in the study of the birational structure of algebraic varieties, as you can read in the very beautiful and modern [2]. The final goal is to explain how the algorithm for the birational classification of varieties works, what steps constitute such algorithm and you will provide a detailed description of the the main ingredients involved and a proof of some of the main results, see [1, 3] – to be determined with me.

Relevant Courses

Essential: Part III Algebraic Geometry, and Algebra.

Useful: Part III Complex Manifolds.

References


104. Pursuit on Graphs .................................................. Professor I. Leader

There has recently been much interest in pursuit questions on graphs. Typically, we have a number of pursuers, working as a team to catch an evader. If this takes place on a graph (so the players live on the vertices of the graph, and in each time-step they move to an adjacent vertex), how many pursuers are needed? And how does this relate to properties of the graph? Most work centres around the conjecture of Meyniel, still unproved, that the number of pursuers need be no more than about the square-root of the number of vertices.

The essay would focus on some results for general graphs, and also on specific cases of interest like random graphs.

Relevant Courses

Essential: None
Useful: Combinatorics

References

[1] Cops and robbers in graphs with large girth and Cayley graphs, P.Frankl
[4] On a generalization of Meyniel’s conjecture on the cops and robbers game, N.Alon and A.Mehrabian

105. Hamiltonian Cycles and Spheres in Hypergraphs .................. Professor I. Leader

The notion of a Hamilton cycle (a cycle through all the vertices) in a graph also makes sense for a hypergraph. There are various versions. One is that we may list the vertices cyclically in such a way that every interval of length $k$ (where the hypergraph consists of $k$-sets) belongs to the hypergraph. Is there an analogue of the well-known Dirac theorem for graphs, which states that if a graph has minimum degree at least $n/2$ then it has a Hamilton cycle?

The aim of the essay is to focus on some classical results on this question, and also on some very recent work on ‘Hamilton spheres’.

References

106. Gravity currents passing over cavities

Professor S. B. Dalziel

High-Reynolds-number gravity currents are found throughout the natural and man-made environments [1] and an understanding of gravity currents provides a good starting point to the nearly identical phenomenon of ‘intrusions’ between two layers of different density. At sufficiently large Reynolds numbers, both gravity currents and intrusions are capable of entraining ambient fluid due to turbulent mixing.

This essay will explore a dense gravity current propagating across a boundary beneath an ambient fluid of uniform density, but for the case where the boundary is interrupted by an isolated cavity filled with a fluid of density greater than that of the ambient. Shear across such a cavity is known to drive mixing and entrainment [2]. This situation is relevant for flows such as a gravity current flowing across the floor of the ocean encountering a depression containing fluid with a greater density, or the urban environment where cold air (from radiative night time cooling) or dense gas is trapped between buildings prior to a cold front crossing the urban environment [3,4].

An essay on this topic will begin with an overview of gravity currents and intrusions before undertaking a review of existing literature relating to this specific problem. From here, the essay could proceed in one (or both) of two directions to assess the impact, propagation and/or mixing associated with the presence of the cavity. Either a set of simple laboratory experiments could be conducted, or shallow water theory could be adapted to provide insight into the phenomenon.

Relevant Courses

Essential: None
Useful: Fluid Dynamics of the Environment

References

107. Clifford algebras and their connection to elementary particle physics . . .
Dr C. Furey

It is no secret that the Clifford algebra $\text{Cl}(1,3)$ underlies Dirac's famous equation. However, this does not mark the end of the known connection between Clifford algebras and elementary particle physics. The Clifford algebra $\text{Cl}(10)$ can furthermore be seen to underlie Spin(10) and SU(5) grand unified theories, in addition to the Pati-Salam model. This essay explores a multitude of ways in which Clifford algebras have been found to be inextricably cemented into our familiar theories of fundamental physics.

Relevant Courses

Essential: Quantum Field Theory
Useful: Symmetries, Fields and Particles; Standard Model

References


Dr. A. Lamacraft

Quantum entanglement, a physical phenomenon in which distinct degrees of freedom are correlated and the quantum state of one particle cannot be written independently of the others, plays an important role in topics such as many-body localization and holography. Recent work has shown that random unitary circuits (RUCs) provide a minimally structured, discrete-time model for entanglement growth in many-body systems [1]. It has been shown analytically and computationally that in one and higher spatial directions, the exponents of operator growth in these systems match those of the Kardar-Parisi-Zhang universality class [2]. This essay will discuss the motivation behind this work, the mathematical background of random unitary circuits, and one or more of the following topics: operator spreading and ways to quantify this (out-of-time-order correlator), the importance of conservation laws in these systems, continuous time analogues to the RUC model [3], and comparison to other models of entanglement growth involving fractons [4] and the SYK model [5].
Relevant Courses

**Essential:** None

**Useful:** Quantum Computation, Statistical Field Theory

References


109. Sheaves on Locales and Internal Locales

Professor P. T. Johnstone

The aim of an essay on this topic would be to develop the theory of sheaves on a locale (or ‘point-free space’) and to describe the notion of internal locale in such a category, with the goal of establishing the equivalence between internal locales in the category of sheaves on $X$ and locales over $X$. For background material on locales, see [1] and [2]; for internal locales, see [3], [4] and [5].

Relevant Courses

**Essential:** Category Theory

References


Conformal Quantum Field Theories in more than two dimensions have become a subject of intense interest in the last ten years. They arise as fixed points at large distances and are relevant in understanding quantum field theories in four dimensions as well as in three for applications in condensed matter physics. CFTs have no mass scales and are expressed in terms of a spectrum of operators with various scaling dimensions and spins. The bootstrap programme, which depends on basic quantum field theory properties such as unitarity and crossing symmetry, provides strong constraints on the possible operators subject to the assumed symmetries of the theory. The numerical bootstrap provides very accurate results for the critical exponents, which depend on the scaling dimensions of various operators, in the three dimensional Ising model. Although much work is numerical there are many analytical results.

The essay should describe the basic framework of CFTs and the motivations for interest in them. It should discuss the essential ideas behind the bootstrap programme. It might consider more specialised topics such as conformal blocks or possibly conformal perturbation theory. Various reviews are contained in [1, 2, 3]. The last is particularly compendious and it is not necessary to digest all of it. A useful background, which predates the recent surge of interest in CFTs, is contained in [4].

References


Stochastic games are a generalisation of Markov decision processes. The aim of the essay would be to understand their equilibria, in the sense of Nash, especially for graphical games. This involves some intricate and fascinating mathematics. The topics to be covered are: Stochastic games as a generalisation of Markov decision processes and repeated games; stochastic graphical games, their properties and computation of the Nash equilibrium; the asymptotic behaviour of these games with respect to their equilibrium points.

Relevant Courses

*Essential:* None

*Useful:* Advanced Probability
References

These can all be found online, or are available from Dr Elliott.

[1] ‘Markov decision processes and stochastic games with total effective payoff’, by Boros, Elbassioni, Gurvich and Makino

112. Stretch factors of pseudo-Anosovs

Dr R. C. H. Webb

The mapping class group Mod(S) of a surface S is the group of homeomorphisms S → S modulo isotopy. The mapping class group occurs naturally in several different fields e.g. as the (orbifold) fundamental group of the moduli space of Riemann surfaces, and, in the study of fibre bundles with fibre S.

An important landmark theorem is the Nielsen–Thurston Classification proved by W. Thurston in the 70s. It states that an element f ∈ Mod(S) of infinite order must either preserve a collection of non-trivial isotopy classes of simple closed curves on S (reducible), or, has a representative homeomorphism which (outside of finitely many singular points) stretches S by λ > 1 in one direction and contracts by 1/λ in another direction (pseudo-Anosov). The number λ is called the stretch factor or dilatation and is a conjugacy invariant of f. Most elements of Mod(S) are pseudo-Anosov, and in our efforts to study them we run into questions in geometry, dynamics, and number theory.

The essay should introduce the notion of a pseudo-Anosov f, give an overview of a proof of the Nielsen–Thurston classification, and explain why the stretch factor λ = λ(f) is an algebraic integer. In fact, Fried showed that λ is a bi-Perron algebraic unit. It is a difficult open problem to determine whether every bi-Perron algebraic unit is the stretch factor of some pseudo-Anosov.

The essay can then finish on one of two different topics. The first choice would be to explain some recent results on the stretch factors of pseudo-Anosovs e.g. each Salem number has a power that is the stretch factor of some pseudo-Anosov. The second choice would be to explain a theorem of W. Thurston, which states that for any Perron number λ there is some outer automorphism of some free group F_n with stretch factor λ.

Relevant Courses

Essential: Algebraic topology
Useful: Differential geometry, Galois theory

References


Our most promising theory for the early universe involves a phase of cosmic inflation, which not only rapidly expands and flattens the universe, but also generates the primordial density perturbations from quantum fluctuations in the inflaton field. While we have good evidence for inflation, e.g. from the Gaussianity, adiabaticity and near-scale invariance of the scalar density perturbations, one prediction of inflation has not yet been found: many inflationary models produce a stochastic background of primordial gravitational waves. A detection of this background would not only provide a definitive confirmation of inflation, but could also give new insights into the microphysics of inflation and, more broadly, physics at the highest energies.

The best current way of finding this gravitational wave background is to search for a characteristic pattern in the polarization of the Cosmic Microwave Background (CMB), the B-mode polarization. This essay should explain the physics underlying the search for this B-mode polarization pattern, which is currently a major area of research in cosmology.

The essay should first review the calculation of the gravitational wave background produced by standard single-field slow-roll inflation, a standard result described in past Part III lecture notes as well as a comprehensive review of the field (Kamionkowski & Kovetz 2016, henceforth KK16). The essay should also explain why the strength of the gravitational wave background (together with the scalar spectral index) can provide powerful constraints on the properties of inflation, such as the potential shape, energy scale, and field excursion (CMB-S4 2016, KK16).

Drawing on KK16, CMB-S4 2016, past lecture notes and other resources, the essay should provide a (brief) review of the basics of CMB polarization, describe what the CMB B-mode polarization is, and explain why it is a powerful probe of inflationary gravitational waves.

The remaining parts of the essay can, to some extent, be tailored to the student’s interests. One option is to explain in detail the major observational challenges in B-mode searches for inflationary gravitational waves, discussing the problems of foregrounds (Bicep/Keck/Planck 2015) and gravitational lensing as well as mitigation methods such as multifrequency cleaning and delensing (Smith et al. 2012). Another option is to focus more on the theoretical background, describing in detail different classes of inflationary models and what these generically predict for B-mode polarization (CMB-S4 2016 and references therein). Students may also discuss a combination of both observational and theoretical aspects.

Relevant Courses

**Essential:** Cosmology

**Useful:** Advanced Cosmology, Quantum Field Theory, General Relativity
Much is known about the behaviour of a turbulent buoyant plume driven by a steady isolated source of buoyancy in a stratified environment where the source of buoyancy is due to either heat or a difference in composition. Indeed, the plume model by Morton, Taylor & Turner [1] is one of the most successful simplified models for a turbulent flow and has been shown to work equally well for laboratory scale flows and flows extending over much of the height of the atmosphere. These ideas are readily extended to the case of bubble plumes [2], where small bubbles alter the bulk density of the fluid. However, differences eventually arise due to the ‘slip velocity’, most noticeably when the rise velocity of the plume becomes comparable with that of an individual bubble. Such bubble plumes occur naturally due to underwater vents and are also utilised in human activities such as the mixing and aeration of stratified lakes [3]. Most previous work, however, has considered a sustained steady source of bubbles.

An essay on this topic will focus on bubble plumes in a stratified environment where the source is generating periodic ‘puffs’ rather than a continuous release of bubbles. The essay should begin with an overview of previous work on steady bubble plumes and brief description of time-dependent phenomena (e.g. [4]) for either bubble or compositional plumes. From here, the essay could proceed in one (or both) of two directions to report on either a set of laboratory experiments, or the relationship between the sequence of puffs and plume theory could be pursued to make some predictions of the behaviour.
115. Strongly interacting gravity currents

S.B. Dalziel

Releases of dense fluid or of light fluid adjacent to a boundary in an otherwise homogeneous environment can lead to the formation of a gravity current [1] that spreads horizontally along the boundary. Such currents occur at high Reynolds number in both the natural and man-made environments. This essay explores if both a dense and light currents, propagating in opposite directions, are released into the same environment. In contrast with the case where both the currents are dense (or both are light) [2], the currents will not collide directly, but unless the environment is very deep, the two currents will still interact, potentially forming a three-layer system.

An essay on this topic will begin with an overview of gravity currents and possibly previous work published on colliding gravity currents. From here, the essay could proceed in one (or both) of two directions to assess the interaction of the currents exploring issues such as their propagation and/or mixing. Either a set of simple laboratory experiments could be conducted, or shallow water theory could be adapted to provide insight into the phenomenon.

Relevant Courses

Essential: None
Useful: Fluid Dynamics of the Environment

References


116. Ultra slow-roll inflation

Dr E. Pajer

In the current standard cosmological model, primordial perturbations are generated during a phase of accelerated expansion called inflation. To reproduce the measured (approximate) scale invariance of these perturbations, it is generally assumed that inflation consisted of a phase of quasi de Sitter expansion. This assumption has the very attractive feature that, on superHubble scales, there always exist a solution for which primordial perturbations are constant in time. This is often called the “adiabatic mode” and matches cosmological observations to sub-percent accuracy. There exist however another possibility, namely that inflation was quite different from de Sitter, and nevertheless still produces an approximate scale invariant spectrum of perturbations. This is realized when the Hubble parameter has a small first derivative but a very large second derivative, and is called Ultra slow-roll (USR) inflation. This model is phenomenologically not very appealing but it offers many theoretical challenges to our understanding. In USR inflation, the adiabatic mode is a subleading correction to a growing and non-adiabatic solution. As a consequence all celebrated single-clock consistency conditions (aka soft theorems) fail in this model. In this essay, one first review the USR inflationary background and the calculation of the power spectrum (refs 6, 1 and 2). Second, one studies the calculation of higher n-point functions in inflation (e.g. from 7 and 8) and the related soft theorems (ref 5, and 8-10).
Relevant Courses

*Essential:* Cosmology, General Relativity, Quantum Field Theory.

*Useful:* Advanced Cosmology.

References


117. **Bounded gaps between primes** ........................................

Dr T. Bloom

One of the oldest problems in mathematics is the Twin Prime Conjecture: that there are infinitely many primes $p, q$ such that $p - q = 2$. This is still unknown, but the past few years have seen a flurry of activity on this problem, beginning with Zhang’s proof in 2013 that there are infinitely many primes whose gaps are bounded by some absolute constant.

In 2014 an alternative proof was given by Maynard, developing sieve theoretic techniques introduced by Goldston, Pintz, and Yildirim. Maynard’s result states that there are infinitely many primes at most 600 apart.

The purpose of this essay is to explain Maynard’s proof, and prove in detail the sieve theoretic aspects of this result. The Bombieri-Vinogradov theorem plays an important role, and so some
discussion and proof of this should also be included. The focus should be on a high-level overview of the sieve techniques used by Goldston, Pintz, Yildirim, and Maynard.

**Relevant Courses**

*Essential:* Analytic Number Theory

**References**
