Faculty of Mathematics
Part III Essays: 2018-19

Titles 1 – 56

Department of Pure Mathematics
& Mathematical Statistics

Titles 57 – 95

Department of Applied Mathematics
& Theoretical Physics

Titles 96 – 133

Additional Essays

Contents

Introductory Notes

General advice. Before attempting any particular essay, candidates are advised to meet the setter in person. Normally candidates may consult the setter up to three times before the essay is submitted. The first meeting may take the form of a group meeting at which the setter describes the essay topic and answers general questions.

Choice of topic. The titles of essays appearing in this list have already been announced in the Reporter. If you wish to write an essay on a topic not covered in the list you should approach your Part III Adviser or any other member of staff to discuss a new title. You should then ask your Director of Studies to write to the Secretary of the Faculty Board, c/o the Undergraduate Office at the CMS (Room B1.28) **no later than 1 February**. The new essay title will require the approval of the Examiners. It is important that the essay should not substantially overlap with any course being given in Part III. Additional Essays will be announced in the Reporter no later than 1 March and are open to all candidates. Even if you request an essay you do not have to do it. Essay titles cannot be approved informally: the only allowed essay titles are those which appear in the final version of this document (on the Faculty web site).

Originality. The object of a typical essay is to give an exposition of a piece of mathematics which is scattered over several books or papers. Originality is not usually required, but often candidates will find novel approaches. All sources and references used should be carefully listed in a bibliography.

Length of essay. There is no prescribed length for the essay in the University Ordinances, but the general opinion seems to be that 5,000-8,000 words is a normal length. If you are in any doubt as to the length of your essay please consult your adviser or essay setter.

Presentation. Your essay should be legible and may be either hand written or produced on a word processor. Candidates are reminded that mathematical content is more important than style. Usually it is advisable for candidates to write an introduction outlining the contents of the essay. In some cases a conclusion might also be required. It is very important that you ensure that the pages of your essay are fastened together in an appropriate way, by stapling or binding them, for example.

Credit. The essay is the equivalent of one three-hour exam paper and marks are credited accordingly.

Final decision on whether to submit an essay. You are not asked to state which papers you have chosen for examination and which essay topic, if any, you have chosen until the beginning of the third term (Easter) when you will be sent the appropriate form to fill in and hand to your Director of Studies. Your Director of Studies should counter-sign the form and send it to the Chairman of Examiners (c/o the Undergraduate Office, Centre for Mathematical Sciences) so as to arrive **not later than 12 noon** of the second Thursday in Easter Full Term, which this year is **Thursday 2 May 2019**. **Note that this deadline will be strictly adhered to.**

Date of submission. You should submit your essay to the Chairman of Examiners (c/o Undergraduate Office, CMS). Your essay should be sent with the completed essay submission form found on page 11 of this document. The form should be completed and signed by you.

Please do not bind or staple the essay submission form to your essay, but instead attach it loosely, e.g. with a paperclip.

Then you should take your essay and the signed essay submission form to the Undergraduate Office (B1.28) at the Centre for Mathematical Sciences so as to arrive **not later than 12 noon** of the second Thursday in Easter Full Term, which this year is **Thursday 2 May 2019**. **Note that this deadline will be strictly adhered to.** If an extension is likely to be needed due to exceptional and unexpected developments, a letter of application and explanation demonstrating the nature of such developments is required from the candidate's Director of Studies. This application should be sent to the Director of Taught Postgraduate Education by the submission date as detailed above. It is expected that such an extension would be (at most) to the following Monday at 12 noon. A student who is dissatisfied with the decision of the Director of Taught Postgraduate Education can request within 7 days of the decision, or by the submission date (extended or otherwise), whichever is earlier, that the Chair of the Faculty review the decision. The provision of any such extenstion will be reported to the examiners for Part III.

Title page. The title page of your essay should bear **ONLY** the essay title. Please **DO NOT** include your name or any other personal details on the title page or anywhere else on your essay.

Signed declaration. The essay submission form requires you to sign the following declaration. It is important that you read and understand this before starting your essay.

> I declare that this essay is work done as part of the Part III Examination. I have read and understood the *Statement on Plagiarism for Part III and Graduate Courses* issued by the Faculty of Mathematics, and have abided by it. This essay is the result of my own work, and except where explicitly stated otherwise, only includes material undertaken since the publication of the list of essay titles, and includes nothing which was performed in collaboration. No part of this essay has been submitted, or is concurrently being submitted, for any degree, diploma or similar qualification at any university or similar institution.

Important note. The *Statement on Plagiarism for Part III and Graduate Courses* issued by the Faculty of Mathematics is reproduced starting on page 12 of this document. If you are in any doubt as to whether you will be able to sign the above declaration you should consult the member of staff involved in the essay. If they are unsure about your situation they should consult the Chairman of the Examiners as soon as possible. The examiners have the power to examine candidates **viva voce** (i.e. to give an oral examination) on their essays, although this procedure is not often used. However, you should be aware that the University takes a very serious view of any use of unfair means (plagiarism, cheating) in University examinations. The powers of the University Court of Discipline in such cases extend to depriving a student of membership of the University. Fortunately, incidents of this kind are very rare.

Return of essays. It is not possible to return essays. You are therefore advised to make your own copy before handing in your essay.

Further advice. It is important to control carefully the amount of time spent writing your essay since it should not interfere with your work on other courses. You might find it helpful to construct an essay-writing timetable with plenty of allowance for slippage and then try your hardest to keep to it.

Research. If you are interested in going on to do research you should, if possible, be available for consultation in the next few days after the results are published. If this is not convenient, or if you have any specific queries about PhD admissions, please contact the following addresses:

Applied Mathematics & Theoretical Physics
research@damtp.cam.ac.uk
DAMTP PhD Admissions,
Mathematics Graduate Office,
Centre for Mathematical Sciences,
Wilberforce Road,
Cambridge CB3 0WA,
United Kingdom.

Pure Mathematics & Mathematical Statistics
research@dpmms.cam.ac.uk
DPMMS PhD Admissions,
Mathematics Graduate Office,
Centre for Mathematical Sciences,
Wilberforce Road,
Cambridge CB3 0WB,
United Kingdom.

MATHEMATICAL TRIPOS, PART III 2019
Essay submission form

To the Chairman of Examiners for Part III Mathematics.

I declare that this essay is work done as part of the Part III Examination. I have read and understood the *Statement on Plagiarism for Part III and Graduate Courses* issued by the Faculty of Mathematics, and have abided by it. This essay is the result of my own work, and except where explicitly stated otherwise, only includes material undertaken since the publication of the list of essay titles, and includes nothing which was performed in collaboration. No part of this essay has been submitted, or is concurrently being submitted, for any degree, diploma or similar qualification at any university or similar institution.

Signed: .... Date: ....

Title of Essay: ....

Essay Number: ....

Name: .... College: ....

Assessor comments:
Your home address is needed to return any essay comments we receive, which will be sent out in June/July 2019. Comments are not mandatory and your assessor may not provide them. Please supply a self-addressed envelope or provide your home address below.

....

....

....

....

....

Spare envelopes are also available in the Undergraduate Office.

Appendix: Faculty of Mathematics Guidelines on Plagiarism

For the latest version of these guidelines please see

http://www.maths.cam.ac.uk/facultyboard/plagiarism/.

University Resources

The University publishes information on *Good academic practice and plagiarism*, including

- a *University-wide statement on plagiarism*;
- Information for students, covering
  - *Your responsibilities*
  - *Why does plagiarism matter?*
  - *Collusion*
- information about *Referencing* and *Study skills*;
- information on *Resources and support*;
- the *University's statement on proofreading*;
- *FAQs*.

There are references to the University statement

- in the Part IB and Part II Computational Project Manuals,
- in the Part III Essay booklet, and
- in the M.Phil. Computational Biology Course Guide.

Please read the University statement carefully; it is your responsibility to read and abide by this statement.

The Faculty Guidelines

The guidelines below are provided by the Faculty to help students interpret what the University Statement means for Mathematics. However neither the University Statement nor the Faculty Guidelines supersede the University's Regulations as set out in the Statutes and Ordinances. If you are unsure as to the interpretation of the University Statement, or the Faculty Guidelines, or the Statutes and Ordinances, you should ask your Director of Studies or Course Director (as appropriate).

What is plagiarism?

Plagiarism can be defined as **the unacknowledged use of the work of others as if this were your own original work**. In the context of any University examination, this amounts to **passing off the work of others as your own to gain unfair advantage**.

Such use of unfair means will not be tolerated by the University or the Faculty. If detected, the penalty may be severe and may lead to failure to obtain your degree. This is in the interests of the vast majority of students who work hard for their degree through their own efforts, and it is essential in safeguarding the integrity of the degrees awarded by the University.

Checking for plagiarism

Faculty Examiners will routinely look out for any indication of plagiarised work. They reserve the right to make use of specialised detection software if appropriate (the University subscribes to *Turnitin Plagiarism Detection Software*). See also the Board of Examinations' statement on How the University detects and disciplines plagiarism.

The scope of plagiarism

Plagiarism may be due to

- **copying** (this is using another person's language and/or ideas as if they are your own);
- **collusion** (this is collaboration either where it is forbidden, or where the extent of the collaboration exceeds that which has been expressly allowed).

How to avoid plagiarism

Your course work, essays and projects (for Parts IB, II and III, the M.Phil. etc.), are marked on the assumption that it is your own work: i.e. on the assumption that the words, diagrams, computer programs, ideas and arguments are your own. Plagiarism can occur if, without suitable acknowledgement and referencing, you take any of the above (i.e. words, diagrams, computer programs, ideas and arguments) from books or journals, obtain them from unpublished sources such as lecture notes and handouts, or download them from the web.

Plagiarism also occurs if you submit work that has been undertaken in whole or part by someone else on your behalf (such as employing a 'ghost writing service'). Furthermore, you should not deliberately reproduce someone else's work in a written examination. These would all be regarded as plagiarism by the Faculty and by the University.

In addition you should not submit any work that is substantially the same as work you have submitted, or are concurrently submitting, for any degree, diploma or similar qualification at any university or similar institution.

However, it is often the case that parts of your essays, projects and course-work will be based on what you have read and learned from other sources, and it is important that in your essay or project or course-work you show exactly where, and how, your work is indebted to these other sources. The golden rule is that **the Examiners must be in no doubt as to which parts of your work are your own original work and which are the rightful property of someone else**.

A good guideline to avoid plagiarism is not to repeat or reproduce other people's words, diagrams or computer programs. If you need to describe other people's ideas or arguments try to paraphrase them in your own words (and remember to include a reference). Only when it is absolutely necessary should you include direct quotes, and then these should be kept to a minimum. You should also remember that in an essay or project or course-work, it is not sufficient merely to repeat or paraphrase someone else's view; you are expected at least to evaluate, critique and/or synthesise their position.

In slightly more detail, the following guidelines may be helpful in avoiding plagiarism.

Quoting. A quotation directly from a book or journal article is acceptable in certain circumstances, provided that it is referenced properly:

- short quotations should be in inverted commas, and a reference given to the source;
- longer pieces of quoted text should be in inverted commas and indented, and a reference given to the source.

Whatever system is followed, you should additionally list all the sources in the bibliography or reference section at the end of the piece of work, giving the full details of the sources, in a format that would enable another person to look them up easily. There are many different styles for bibliographies. Use one that is widely used in the relevant area (look at papers and books to see what referencing style is used).

Paraphrasing. Paraphrasing means putting someone else's work into your own words. Paraphrasing is acceptable, provided that it is acknowledged. A rule of thumb for acceptable paraphrasing is that an acknowledgement should be made at least once in every paragraph. There are many ways in which such acknowledgements can be made (e.g. "Smith (2001) goes on to argue that ..." or "Smith (2001) provides further proof that ..."). As with quotation, the full details of the source should be given in the bibliography or reference list.

General indebtedness. When presenting the ideas, arguments and work of others, you must give an indication of the source of the material. You should err on the side of caution, especially if drawing ideas from one source. If the ordering of evidence and argument, or the organisation of material reflects a particular source, then this should be clearly stated (and the source referenced).

Use of web sources. You should use web sources as if you were using a book or journal article. The above rules for quoting (including 'cutting and pasting'), paraphrasing and general indebtedness apply. Web sources must be referenced and included in the bibliography.

Collaboration. Unless it is expressly allowed, collaboration is collusion and counts as plagiarism. Moreover, as well as not copying the work of others you should not allow another person to copy your work.

Links to University Information

- Information on *Plagiarism and good academic practice*, including
  - *Students' responsibilities*.
  - *Information for staff*.

Table 1: **A Timetable of Relevant Events and Deadlines**

Friday 1 February	Deadline for Candidates to request additional essays.
Thursday 2 May, noon	Deadline for Candidates to return form stating choice of papers and essays.
Thursday 2 May, noon	Deadline for Candidates to submit essays.
Thursday 30 May	Part III Examinations begin.

Comments. If you feel that these notes could be made more helpful please write to *The Chairman of Examiners, c/o the Undergraduate Office, CMS.*

Further information. Professor T.W. Körner (DPMMS) wrote an essay on Part III essays which may be useful (though it is slanted towards the pure side). It is available via his home page

https://www.dpmms.cam.ac.uk/~twk/Essay.pdf

1. The Congruence Subgroup Problem ....
Professor E. F. J. Breuillard

This is a famous group problem in group theory aiming at understanding finite index subgroups of arithmetic groups. Mennicke and Bass-Milnor-Serre proved that every finite index subgroup of $SL_n(\mathbf{Z})$, $n \geq 3$ contains a congrence subgroup, namely the kernel of the reduction modulo N homomorphism $SL_n(\mathbf{Z}) \to SL_n(\mathbf{Z}/N\mathbf{Z})$ for some integer N. This question can be formulated more generally for an arbitrary arithmetic group and a conjecture of Serre describes what is expected.

The essay would focus on the proof for $SL_n(\mathbf{Z})$ first and then venture to other territories at the student's discretion.

Relevant Courses

Essential: None

Useful: Part II: Representation theory, Part III: Algebra, Graduate: Discrete subgroups of Lie groups.

References

[1] A. Lubotzky and D. Segal, Subgroup growth. Birkhuser. (2003).

[2] V. Platonov, A. Rapinchuk, Algebraic groups and number theory. Pure and Applied Mathematics. 139. Academic Press. (1994).

[3] B. Sury, The congruence subgroup problem. Hindustan book agency. (2003).

[4] H. Bass, J. Milnor, J-P. Serre, Solution of the congruence subgroup problem for SL_n ($n \geq 3$) and Sp_{2n} ($n \geq 2$). Publications mathmatiques de lI.H..S., 33 (1967), p. 59–137.

2. Non-Arithmetic Lattices ....
Professor E. F. J. Breuillard

A lattice in a Lie group is a discrete group of finite co-volume. A celebrated theorem of Margulis asserts that every lattice in a $SL_n(\mathbf{R})$, $n \geq 3$, is arithmetic. More generally every irreducible lattice in a semisimple Lie group of rank at least 2 is arithmetic. This was extended by Corlette and Gromov-Schoen to all rank one Lie groups, except for the families $SO(n,1)$ and $SU(n,1)$, which correspond respectively to groups of isometries of real and complex hyperbolic spaces. Non-arithmetic lattices in $SO(n,1)$ have been constructed as certain reflection groups for certain n and a general construction has been given by Gromov and Piatetski-Shapiro. In $SU(n,1)$, $n \geq 2$ only finitely many examples are known of non-arithmetic lattices and none are known for $n \geq 4$.

The essay would aim at giving an exposition of the Gromov and Piatetiski-Shapiro examples in the first place, and then move to the Mostow examples of non-arithmetic lattices in $SU(2,1)$ and beyond if time permits.

Relevant Courses

Essential: Part II: Differential Geometry

Useful: Part III: Algebraic topology, Graduate: Discrete subgroups of Lie groups.

References

[1] M. Gromov and I. Piatetski-Shapiro, Non-arithmetic groups in Lobachevsky spaces, Publications mathématiques de l' I.H.E.S., tome 66 (1987), p. 93-103

[2] D. Mostow, On a remarkable class of polyhedra in complex hyperbolic space, Pacific Journal of Math. (1980).

[3] D. Witte Morris, Introduction to arithmetic groups, Book available on the author's webpage.

[4] P. Deligne and G. Mostow, Monodromy of hypergeometric functions and non-lattice integral monodromy. Publications mathématiques de l' I.H.E.S., 63:5–89, (1986).

3. Canonical Kähler Metrics on Projective Varieties .... Dr R. Dervan

The natural candidate for a "best" metric on a smooth projective variety is a Kähler metric with constant scalar curvature. The existence problem for such metrics is subtle, with the guiding principle being Donaldson's conjecture that the existence of such metrics should be equivalent to the algebro-geometric notion of K-stability. The goal of this essay is to discuss Donaldson's proof of one direction of this conjecture: the existence of a constant scalar curvature Kähler metric implies K-semistability [1].

The essay should begin with the definition of constant scalar curvature Kähler metrics, as given in [2]. The essay should next discuss the Hilbert polynomial and weight polynomials of projective varieties, including a proof that these are indeed polynomials (proven for example in [3, Theorem 9.1 and Proposition 3.12]). Next the essay should define K-semistability following [1]. The bulk of the essay should consist of proving [2, Theorem 1]. Some parts of Donaldson's proof are only sketched, and the essay should contain a clear account of the claim that the function $f(t)$ used in [2, p464] is increasing; one exposition of this is [2, Lemma 7.19]. The proof of [2, Proposition 3] given by Donaldson is technically challenging and an alternative proof can be found in [4, Theorem 27]. The expansion of the density of states function ρ_k used by Donaldson should be stated clearly, but the proof of this expansion should be left as a black box.

An ambitious author may wish to give an example of a K-unstable variety, for example through the construction given in [2, Section 6.5].

Relevant Courses

Essential: Algebraic Geometry, Complex Manifolds

References

[1] S. K. Donaldson *Lower bounds on the Calabi functional.* J. Differential Geom. Volume 70, Number 3 (2005), 453-472.

[2] G. Székelyhidi *An introduction to extremal Kähler metrics.* Graduate Studies in Mathematics, 152. American Mathematical Society, Providence, RI, 2014. xvi+192, a shorter version also available at https://www3.nd.edu/~gszekely/notes.pdf

[3] Boucksom, S., Hisamoto, T. and Jonsson, M. *Uniform K-stability, Duistermaat-Heckman measures and singularities of pairs.* Ann. Inst. Fourier (Grenoble) 67 (2017), no. 2, 743-841.

[4] Wang, X. *Moment map, Futaki invariant and stability of projective manifolds.* Comm. Anal. Geom. 12 (2004), no. 5, 1009-1037.

4. Modular Curves and the Class Number One Problem ....
Dr T. A. Fisher

The class number one problem of Gauss asks for the complete list of imaginary quadratic fields with class number one. It has long been known that there are at least nine such fields. The first proofs that this list is complete were given by Baker (using linear forms in logarithms) and Stark (using modular functions) in 1966. This essay should describe the latter approach, which is related to earlier work of Heegner. The starting point should be the treatment of Serre [6, Sections A.5 and A.6], who reduces the problem to that of determining the integral points on a certain modular curve $X^+_{\mathrm{ns}}(N)$. Nowadays the proof can be completed in several different ways, that is, by considering different values of N; see [1], [2], [4] and [5].

Relevant Courses

Essential: None

Useful: Elliptic Curves, Algebraic Number Theory

References

[1] B. Baran, A modular curve of level 9 and the class number one problem, *J. Number Theory* 129 (2009), no. 3, 715–728.

[2] I. Chen, On Siegel's modular curve of level 5 and the class number one problem, *J. Number Theory* 74 (1999), no. 2, 278–297.

[3] D. A. Cox, *Primes of the form* x^2+ny^2*, Fermat, class field theory and complex multiplication*, Wiley, 1989.

[4] M. A. Kenku, A note on the integral points of a modular curve of level 7, *Mathematika* 32 (1985), no. 1, 45–48.

[5] R. Schoof and N. Tzanakis, Integral points of a modular curve of level 11, *Acta Arith.* 152 (2012), no. 1, 39–49.

[6] J.-P. Serre, *Lectures on the Mordell-Weil theorem*, Aspects of Mathematics, Friedr. Vieweg & Sohn, Braunschweig, 1997.

5. Classical Invariant Theory and Moduli of Genus 2 Curves ....
Dr T. A. Fisher

This essay should begin by reviewing the classical (i.e. 19th century) invariant theory of binary forms of degree n with particular reference to the cases $n = 4$ and $n = 6$. The case $n = 4$ is related to elliptic curves (see [7]), which are classified up to isomorphism (over an algebraically closed field) by their j-invariant. The case $n = 6$ leads to the definition of the Igusa (or Igusa-Clebsch) invariants that likewise classify genus 2 curves. The main aim of the essay should be to describe the algorithm of Mestre [4] for recovering the equation for a genus 2 curve from its Igusa invariants. If time and space permit, then the connection to Siegel modular forms, or applications such as those in [6], could be discussed.

Relevant Courses

Essential: None

Useful: Elliptic Curves, Algebraic Geometry

References

[1] J.H. Grace and A. Young, *The algebra of invariants*, CUP, 2010.

[2] D. Hilbert, *Theory of algebraic invariants*, CUP, 1993.

[3] J. Igusa, Arithmetic variety of moduli for genus two, *Ann. of Math.* (2) 72, (1960), 612–649.

[4] J.-F. Mestre, Construction de courbes de genre 2 à partir de leurs modules, in *Effective methods in algebraic geometry*, T. Mora and C. Traverso (eds), Birkhäuser, 1991.

[5] P.J. Olver, *Classical invariant theory*, CUP, 1999.

[6] X.D. Wang, 2-dimensional simple factors of $J_0(N)$, *Manuscripta Math.* 87, (1995), no. 2, 179–197.

[7] A. Weil, Remarques sur un mémoire d'Hermite, *Arch. Math.* (Basel) 5, (1954), 197–202.

6. Fraenkel-Mostowski Models for Set Theory .... Dr T. Forster

Fraenkel-Mostowski models were developed by Fraenkel, Mostowski and Specker, originally as a means of demonstrating the independence of the Axiom of Choice from the other axioms of set theory. Later refinements were able to tease apart various choice-like principles: for example one can show that none of the implications in the chain: AC $\rightarrow$ "Every partial order can be refined to a total ordering" $\rightarrow$ "every set can be totally ordered" $\rightarrow$ "Every set of finite sets has a choice function" can be reversed.

In each case the sets of an FM model are sets that are in a suitable sense invariant under the action of a judiciously chosen group of permutations of atoms; so there is a bit of group theory involved, and of course a bit of Set Theory. These same ideas of invariance are of course in play in the "forcing" proofs of independence exhibited later by Cohen (and which are touched on in Part III "Topics in Set Theory") but the focus there is on the forcing and the purpose of this essay is rather to study the invariance.

There is a variety of applications/aspects which the student can choose between. There are the independence proofs of course; there are recent developments in Set Theory using FM techniques (without forcing); there is a nice semantics for capture-avoiding substitution (google "Fresh ML"), and the corpus of FM work cries out for an abstract general treatment—and such a project might appeal to students who are inclined towards Category theory.

Relevant Courses

Essential: Part II Logic and Set Theory or equivalent.

Useful: Undergraduate Topology; Part III Category theory; Part III Topics in Set Theory.

References

Jech, The Axiom of Choice, Dover would be a good place to start looking. A more detailed reading list can be obtained from the Essay Sponsor on application

7. The Probability that a Random Matrix is Singular .... Professor W. T. Gowers

Let A be a random $n \times n$ matrix with ± 1 entries. What is the probability that A is singular? One way that it can be singular is if two of its rows are equal or add to zero, and the probability that this happens is approximately $n(n-1)2^{-n}$. It is conjectured that this possibility is the main way that singularity of A occurs: that is, there is a conjectured upper bound of $(1+o(1))n^2 2^{-n}$.

This seems to be a hard conjecture. Even proving that the probability tends to zero with n was a significant achievement, due to Komlós, and obtaining an exponential upper bound was a further breakthrough, due to Kahn, Komlós and Szemerédi. Their bound was $O(0.999^n)$.

The purpose of this essay is to present a proof of an upper bound of $(3/4 + o(1))^n$, which was proved by Tao and Vu. Rather surprisingly, their proof involved tools from additive combinatorics, including Freiman's theorem: a key aim of the essay should be to make it clear to the reader why Freiman's theorem has any relevance to the problem.

Relevant Courses

Essential: None

Useful: Introduction to Discrete Analysis would certainly help, since results closely related to Freiman's theorem will be proved there. However, this is more like useful background information than an essential prerequisite: in principle it would be possible to do the essay just understanding the statement of Freiman's theorem.

References

[1] Terence Tao and Van Vu, On the singularity probability of random Bernoulli matrices, https://arxiv.org/pdf/math/0501313.pdf

[2] Jeff Kahn, János Komlós and Endre Szemereédi, On the probability that a random ±1 matrix is singular, http://users.uoa.gr/~apgiannop/matrices/Kahn_Komlos_Szemeredi_1995.pdf

[3] This talk by Van Vu might be helpful for the big picture: https://www.microsoft.com/en-us/research/video/singularity-of-random-bernoulli-matrices/

8. Probabilistically Checkable Proofs .... Professor W. T. Gowers

Unless P=NP, which most people believe is not the case, there is no polynomial-time algorithm for determining whether a graph contains a clique of a given size. However, there is a polynomial-time algorithm for checking whether a given set of vertices spans a clique: one just checks that all the pairs of vertices in the set are joined by edges.

Suppose that you did not insist on 100% certainty that a graph contained a clique, but merely on 99.999% certainty. A major, and very surprising, result in theoretical computer science is the PCP theorem, which roughly speaking states that there is a way of converting a graph into a string of bits in polynomial time such that if you look at a constant number of those bits at random, then you can do a test that will always fail if the graph does not contain a clique and will pass with probability at least 1/2 if it does contain a clique. By repeating the test, this 1/2 can be boosted as close as you like to 1. The first proof of this theorem was very long and complicated, but more recently, in another important development, Irit Dinur found a much more accessible proof. The main purpose of this essay is to present that proof. A secondary purpose is to explain the relationship between the PCP theorem and results that say that if P≠NP, then certain computational problems are not just hard to solve exactly, but hard even to solve approximately. For example, there is no polynomial-time algorithm that outputs "yes" if a graph contains a clique of size m_1 and "no" if it doesn't contain a clique of size m_2, even when m_2 is much smaller than m_1.

Relevant Courses

None, but the proof is combinatorial in flavour.

References

[1] Irit Dinur, The PCP theorem by gap amplification, available online at http://www.wisdom.weizmann.ac.il/~dinuri/mypapers/combpcp.pdf

9. Riemann-Hilbert Correspondence for Differential Equations with Regular and Irregular Singularities . . . . Professor I. Grojnowski

The aim of this essay is to understand the basics of the Riemann Hilbert correspondence, which is about solutions of linear differential equations in many variables.

It will be useful if you care about non-commutative geometry and mirror symmetry, number theory, or representation theory and the Langlands programme. It might also be useful if you care about differential equations.

An extremely ambitious essay would include an account of the theorem of Mochizuki and Kedlaya. An account of the paper of Katz would still be a very good essay.

Begin with the basics of vector bundles with flat connection on a projective algebraic variety that are allowed regular singularities along a normal crossing divisor.

The one dimensional version of this is the 19th century theory of Fuchsian differential equations; the higher dimensional version is a theorem of Deligne's. You can read about this in the articles of Haefliger and Malgrange in Borel's book.

Then learn the theory of irregular singularities, and Stokes structures, again in the one dimensional case. Kac's article is a good source, as is Malgrange's books and papers. (There are also classic textbooks on differential equations...)

To understand the statement and proof of the higher dimensional versions, you will need some notions from birational algebraic geometry. You may also wish to learn about D-modules.

If you are interested in this essay, we should discuss the best way in to the subject for you.

The eventual goal is to read and understand the papers of Sabbah, Mochizuki or Kedlaya, and the consequences of this.

References

Many textbook expositions of D-modules now exist. The two best are by the originators of the subject—Kashiwara and Bernstein (the latter are printed notes, available on the web somewhere).

Borel's book, Algebraic D-modules, is not a good place to read the material, except for the articles by Haefliger and Malgrange, which are excellent. Katz's article is

Katz, N. M. Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin. Publications Mathmatiques de L'Institut des Hautes Scientifiques 39, 175-232 (1970).

The following articles will be *completely* incomprehensible to you to start — understanding them is the end goal. Don't browse them and be put off!

Kashiwara, M. Riemann-Hilbert correspondence for irregular holonomic D-modules. Jpn. J. Math. 11, 113149 (2016).

K.S. Kedlaya, Good formal structures for flat meromorphic connections. I: Surfaces, Duke Math. J., 154 (2010), 343418.

K.S. Kedlaya, Good formal structures for flat meromorphic connections. II: Excellent schemes, J. Amer. Math. Soc., 24 (2011), 183229.

T. Mochizuki, Good formal structure for meromorphic flat connections on smooth projective surfaces, In: Algebraic Analysis and Around, Adv. Stud. Pure Math., 54, Math. Soc. Japan, Tokyo, 2009, pp. 223253.

10. Quantum Groups, KZ Equations, $Gal(\bar{Q}/Q)$, Periods .... Professor I. Grojnowski

In the mid 1980s Drinfeld and Jimbo defined quantum groups, algebras U_h over $\mathbb{C}[[h]]$ which are deformations of the enveloping algebra U_0 of a semisimple Lie algebra.

More precisely, U_h is a Hopf algebra, free over $\mathbb{C}[[h]]$, such that U_h/hU_h is the enveloping algebra. Now, it is easy to see (and you will, in this essay!), that enveloping algebras of semisimple Lie algebras cannot deform — U_h is isomorphic to $U_0 \otimes \mathbb{C}[[h]]$, so one can think of this as saying what is actually changing is how you make the tensor product of two U_h-modules a U_h-module.

One way of doing this, invented by Drinfeld, is to change the associativity constraint, that is the isomorphism $V_1 \otimes (V_2 \otimes V_3) \simeq (V_1 \otimes V_2) \otimes V_3$ between the tensor product of three modules. And one way of doing this is to use the monodromy of the Knihzik-Zamalodchikov equation, an explicit flat connection on vector bundles over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

The first main goal of this essay is to understand this precisely — that is, to understand Drinfeld's theorem computing the deformations of U_0 as a bialgebra, the construction of the KZ associator, and the Drinfeld-Kohno theorem.

This can all be found in the original papers, or in the lovely textbook by Etingof-Schiffmann.

You may then continue in various ways. You can learn more about the KZ-equation, the connection with unipotent local systems, special values of mutli-zeta functions, and the Galois group of $\bar{\mathbb{Q}}/\mathbb{Q}$.

Or you may prefer to learn about more recent approaches to deformation theory — Kontsevich's theorem, and its various proofs, and then the topologist's hierarchy of homotopy commutative algebras, the E_n-algebras.

Or you may prefer to understand properties of quantum groups which are not explained by deformation theory — the quantum groups are defined over $\mathbb{Z}[q, q^{-1}]$, not just the formal completion of this at $q = e^h = 1$.

References

[1] Etingof, Pavel; Schiffmann, Olivier. Lectures on Quantum Groups. *International Press,* Somerville, MA, 2002.

[2] Brown, Francis. Motivic Periods and the Projective Line Minus Three Points, *Proceedings of the ICM* 2014, arXiv:1407.5165

[3] Drinfeld, V.G. On Quasitriangular Quasi-Hopf Algebras and on a Group that is Closely Connected with Gal(Q/Q). *Leningrad Math. J. 2* (1991), no. 4, 829—860

[4] Drinfeld, V.G. Quasi-Hopf Algebras. *Leningrad Math. J. 1* (1990), no. 6, 1419—1457

11. The Foundations of Logarithmic Geometry . . . . Professor M. Gross

The theory of logarithmic schemes was developed in the 1980s by Illusie-Fontaine and Kazuya Kato. As described by Kato, a logarithmic structure on a scheme is a "magic powder" which makes relatively nice singular schemes look smooth. A typical example is a normal crossings divisor, which formally looks smooth if viewed as a log scheme.

While the original motivation for introducing log schemes was for its arithmetic applications, more recently log schemes have found powerful applications in mirror symmetry. This essay should cover the fundamentals of log geometry, and then explore applications of interest to the essay writer.

The original papers [1], [2] are dense but readable. Chapter 3 of [3] contains a more relaxed introduction to parts of the theory needed for mirror symmetry. [4] is a partial encyclopedic manuscript on the subject. These sources are more than enough to get started. Further avenues can be explored once the basics are mastered, following up in the mirror symmetry direction via [3] and references therein, or in the arithmetic direction.

Relevant Courses

Essential: Part III Algebraic Geometry

References

[1] Luc Illusie, "Logarithmic spaces (according to K. Kato), Barsotti Symposium in Algebraic Geometry (Abano Terme, 1991), Perspec. Math., Vol 15, Academic Press, San Diego, CA 1994. pp. 183–203.

[2] Kazuya Kato, "Logarithmic structures of Fontaine-Illusie," Algebraic Analysis, geometry, and number theory (Baltimore, MD 1988), Johns Hopkins Univ. Press, Baltimore, MD, 1989, pp. 191-224.

[3] Mark Gross, *Tropical geometry and mirror symmetry*, CBMS Regional Conference Series in Mathematics, **114**. American Mathematical Society, Providence, RI, 2011. xvi+317 pp

[4] Arthur Ogus, *Lectures on Logarithmic Algebraic Geometry*, available at http://www.math.berkeley.edu/~ogus

12. Algebraic Stacks .... Professor M. Gross

Algebraic stacks are a vast generalization of the notion of scheme, developed partly to describe various moduli spaces. For example, $\mathcal{M}_g$, the moduli space of algebraic curves of genus g, cannot be described as a scheme, but is what is known as a Deligne-Mumford stack. Morally, this is a geometric object which is locally a quotient of a scheme by a finite group, but the geometric object remembers something about this local description. If the scheme is smooth, then we obtain the algebraic-geometric equivalent of an orbifold. More generally, an Artin (or algebraic) stack allows quotients by much more complicated equivalence relations. For example, the trivial action of an algebraic group G on the point has a well-defined quotient in the world of algebraic stacks, and this quotient plays the role of the classifying space BG in algebraic geometry. See [2] for a very brief survey, and [3] for a longer survey.

This essay would involve internalizing the (very complicated) definition of stacks, and giving some application(s). The most obvious application is the construction of the moduli space of stable curves [1]. Other possibilities include the construction of the Chow group for Artin stacks [5], and Artin's criterion for algebraicity of stacks [4]. The former will require delving into the theory of algebraic cycles, the latter into deformation theory.

There are several sources for the definitions. The original papers [1] and [4] give concise definitions, and [6] covers these in a more expansive way (but is in french). There are a number of online resources, (follow the links from the wikipedia page on stacks) and the Stacks Project [8], the latter being a vast compendium of most of algebraic geometry and probably not so useful for a beginner. There is also a good new book on the subject by Martin Olsson, [7].

Relevant Courses

Necessary: Part III Algebraic Geometry (Michaelmas term).

References

[1] Deligne, P., Mumford, D. *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. No. **36**, 1969, 75–109.

[2] Edidin, D., *What is a... stack?*, Notices of the AMS, April 2003, 458–459.

[3] Fantechi, B., *Stacks for everybody,* European Congress of Mathematics, Vol. I (Barcelona, 2000), 349–359, Progr. Math., 201, Birkhäuser, Basel, 2001.

[4] Artin, M., *Versal deformations and algebraic stacks*, Inventiones Math., **27**, 165–189.

[5] Kresch, A., *Cycle groups for Artin stacks,* Invent. Math. **138** (1999), 495–536.

[6] Laumon, G.; Moret-Bailly, L., *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics **39**. Springer-Verlag, Berlin, 2000. xii+208 pp.

[7] Olsson, M., *Algebraic spaces and stacks*, AMS Colloquium Publications, Volume 62, 2016.

[8] http://stacks.math.columbia.edu/

13. Tropical Geometry .... Professor M. Gross

Tropical geometry is algebraic geometry over the so-called tropical semiring, the semiring of real numbers with addition being maximum and multiplication being ordinary addition. Thus a "tropical polynomial" in n variables is really a function given as a maximum of a collection of affine linear functions with integral slopes. The "zero locus" of such a function is interpreted as the locus where such a function isn't linear. For example, we define a tropical hypersurface in $\mathbf{R}^n$ as the non-linear locus of such a function.

While these resulting objects are very combinatorial in nature, there turns out to be a rich and surprising relationship between tropical geometry and complex geometry. For example, Mikhalkin [1] really started the subject by showing that curves in the complex projective plane can be counted tropically.

There is now a wide literature in the subject, and some of this literature does not require very much background in algebraic geometry. Thus this essay should be accessible to students who have not taken the Part III Algebraic Geometry class, as long as they are willing to learn a little bit of algebraic geometry on the way.

Chapter 1 of [2] gives an introduction to tropical geometry, and Chapter 4 gives the most technologically advanced proof of Mikhalkin's result. [3] gives a very elementary introduction to the subject, and [4] gives an elementary, completely combinatorial proof (but see Chapter 2 of [2] for the necessary background). Depending on the tastes of the essay writer, various aspects of the theory can be explored. Possibilities include (a) the relationship between tropical geometry and amoebas [5]; (b) enumerative applications [1], [4]; (c) applications to mirror symmetry, especially [2], Chapter 5.

Relevant Courses

Essential: None

Useful: Part III Algebraic Geometry (Michaelmas term).

References

[1] Grigory Mikhalkin, "Enumerative tropical geometry in $\mathbf{R}^2$," arXiv:math/0312530.

[2] Mark Gross, *Tropical geometry and mirror symmetry*, CBMS Regional Conference Series in Mathematics, **114**. American Mathematical Society, Providence, RI, 2011. xvi+317 pp

[3] J. Richter-Gebert, B. Sturmfels, T. Theobald, "First steps in tropical geometry," arXiv:math/0306366.

[4] Andreas Gathmann and Hannah Markwig, "Kontsevich's formula and the WDVV equations in tropical geometry," Advances in Mathematics, 217, (2007), 537–560.

[5] Grigory Mikhalkin, "Amoebas of algebraic varieties," arXiv:math/0108225.

14. Classifying Toposes .... Professor J. M. E. Hyland

Any Grothendieck Topos can be regarded as the classifying topos for some (by no means unique) geometric theory. There is an early account of the basic ideas in [4] and a somewhat later one from a different point of view in [3]. Peter Johnstone's treatment in [2] is more sophisticated, placing classifying toposes within a general development of Topos Theory. In all three references classifying toposes occur quite late and as that suggests a good deal of both categorical and logical background is required for a full appreciation of the ideas. However those taking both the courses in Category Theory and Model Theory could certainly contemplate an essay. The task is made a little less intimidating by the existence of [1], written by Johnstone's student, Olivia Caramello. That gets pretty directly to the subject though the background assumed is more subtle than may at first appear.

An essay could approach the topic from a number of different directions. One the one hand there is a good deal of fundamental theory which could be spelt out. Then again one could focus on ways of showing that particular toposes (however given) classify specific theories. (Examples of classifying toposes can be found in the references but one might well look for others.) More ambitiously the theory can be developed relative to a more general base topos; or one might try to treat recent developments regarding the Completeness Theorem. Anyone contemplating an essay on the topic is advised to discuss possibilities at an early stage.

Relevant Courses

Essential: Category Theory

Useful: Model Theory

References

[1] O. Caramello. *Toposes, Sites, Theories.* OUP 2018.

[2] P. T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium.* Volumes 1 and 2. OUP 2002.

[3] S. Mac Lane and I. Moerdijk. *Sheaves in geometry and logic: a first introduction to topos theory.* Springer-Verlag, 1992.

[4] M. Makkai and G. Reyes. *First-order categorical logic.* Lecture Notes in Mathematics 611, Springer-Verlag 1977.

15. Locally Presentable and Accessible Categories .... Professor P. T. Johnstone

Locally presentable categories were introduced by Gabriel and Ulmer [1,2], and were an early attempt to capture the essential categorical structure of the category of models of a theory. The fact that they succeeded in doing just this, for a particular (very natural) class of 'essentially algebraic' theories, was proved by M. Coste [3]. More recently, attention has focused on the much larger class of accessible categories [4,5], which are categories of models of theories in a much broader sense; locally presentable categories are precisely those accessible categories which are complete as categories. An essay on this topic could either take as its goal the main theorem characterizing accessible categories as categories of models, or it could survey the way in which particular properties of the axiomatization of a theory are reflected in properties of its category of models. (Some examples of the latter may be found in [6].)

Relevant Courses

Essential: Category Theory

References

[1] F. Ulmer, Locally α-presentable and locally α-generated categories, in *Reports of the Midwest Category Seminar V*, Lecture Notes in Math. vol. 195 (Springer–Verlag, 1971), 230–247. (This is a summary in English of the main results of [2].)

[2] P. Gabriel and F. Ulmer, *Lokal präsentierbare Kategorien*, Lecture Notes in Math. vol. 221 (Springer–Verlag, 1971).

[3] M. Coste, Localisation, spectra and sheaf representation, in *Applications of Sheaves*, Lecture Notes in Math. vol. 753 (Springer–Verlag, 1979), 212–238.

[4] M. Makkai and R. Paré, *Accessible Categories: the Foundations of Categorical Model Theory*, Contemporary Math. vol. 104 (Amer. Math. Soc., 1989).

[5] J. Adámek and J. Rosický, *Locally Presentable and Accessible Categories*, L.M.S. Lecture Notes Series no. 189 (C.U.P., 1994).

[6] P.T. Johnstone, *Sketches of an Elephant: a Topos Theory Compendium*, chapter D2, Oxford Logic Guides 44 (O.U.P., 2002), 861–889.

16. Synthetic Differential Geometry ....
Professor P. T. Johnstone

In 1967, F.W. Lawvere suggested that the traditional analytic approach to differential geometry might be replaced by a ‘synthetic’ approach, in which one would begin by directly axiomatizing (a category containing) the category of smooth manifolds. Lawvere’s axioms are incompatible with classical logic, and thus with the traditional conception of what a smooth manifold is: it was not until the development of elementary topos theory in the 1970s that it became possible to give explicit models for them. An essay on this topic could either concentrate on developing the axiomatics (for which Anders Kock’s first book [1] is probably still the best introduction, although Kock’s later book [2] and René Lavendhomme’s [3] are also recommendable); or, more ambitiously, it could describe the construction of a ‘well-adapted’ model of the axioms, in which the classical category of manifolds is nicely embedded. (The latter would require the development of a good deal of topos theory; suitable references would include [4] and [5].)

Relevant Courses

Essential: Category Theory

References

[1] A. Kock, *Synthetic Differential Geometry*, Cambridge University Press (L.M.S. Lecture Notes series), 1981 (second edition 2006).

[2] A. Kock, *Synthetic Geometry of Manifolds*, Cambridge University Press (Cambridge Tracts in Mathematics), 2010.

[3] R. Lavendhomme, *Basic Concepts of Synthetic Differential Geometry*, Kluwer, 1996.

[4] I. Moerdijk and G.E. Reyes, *Models for Smooth Infinitesimal Analysis*, Springer–Verlag, 1991.

[5] P.T. Johnstone, ‘Synthetic Differential Geometry’, chapter F1 of *Sketches of an Elephant: a Topos Theory Compendium* (not published yet, but preprint copies available from the author).

17. Lagrangian Tori in $\mathbb{R}^6$ ....

Dr A. M. Keating

Lagrangian submanifolds are distinguished half-dimensional submanifolds of symplectic manifolds – for instance, in the case of $\mathbb{C}^n$, the tori $\{(z_1, \ldots, z_n) : |z_i| = a_i\}$, for some positive constants a_i. The nicest condition that one can impose on a closed Lagrangian submanifold in $\mathbb{C}^n$ is for it to be monotone; in the aforementioned example, this amounts to requiring that all of the a_i be equal. Up to suitable notions of equivalence, there is a unique (automatically monotone) Lagrangian circle in $\mathbb{C}$. In $\mathbb{C}^2$, it is widely expected that there are two. The goal of this essay is to give an account of a beautiful result of Auroux, who, in contrast, produced an infinite collection of monotone Lagrangian tori in $\mathbb{C}^3$.

This essay should readily build on parts of the Symplectic Geometry course. After briefly recalling relevant definitions from the course, the essay should start by explaining Auroux' construction from [1]; you may find it helpful to understand the perspective of Section 5 of [1], which draws on constructions in [2]. To tell the different tori apart, Auroux uses an invariant which comes from counting certain pseudo-holomorphic discs; the essay should proceed to give an account of this. You may choose to treat various amounts of Floer-theoretic background as a 'black-box'.

Relevant Courses

Essential: Differential Geometry; Symplectic Geometry; basic notions from Algebraic Topology

Useful: Algebraic Geometry

References

[1] D. Auroux, *Infinitely many monotone Lagrangian tori in* $\mathbb{R}^6$, Invent. Math., 201(3):909–924, 2015 (arXiv:1407.3725)

[2] Y. Lekili, M. Maydanskiy, *The symplectic topology of some rational homology balls*, Comm. Math. Helv. 89 (2014), 571596 (arXiv:1202.5625).

For the symplectic geometry background (to be covered in the Lent term course):

[4] D. McDuff and D. Salamon, *Introduction to symplectic topology*, 3rd edition, Oxford University Press, 2017

[5] A. Cannas da Silva, *Lectures on Symplectic Geometry*, Springer, 2nd edition (2008), also available at *https://people.math.ethz.ch/∼acannas/*

18. Complements of Hyperplane Arrangements ....

Dr A. M. Keating

A hyperplane arrangement is a finite collection of affine hyperplanes in $\mathbb{C}^n$. These have been the object of considerable research, notably regarding the topological properties of their complements in $\mathbb{C}^n$. The goal of this essay is to study some of these properties. It should begin by discussing the fundamental group of the complement of a hyperplane arrangement, with starting point the Zariski–Van Kempen theorem. Several directions are then possible, for instance: Hattori's result on the topology of the complement of a generic arrangement; Deligne's proof that a simplicial arrangement gives a $K(\pi, 1)$ Eilenberg-MacLane space; the description of the cohomology ring of the complement in terms of generators and relations.

Relevant Courses

Essential: Algebraic Topology

Useful: Algebraic Geometry, Differential Geometry

References

[1] P. Orlik, H. Terao, *Arrangements of hyperplanes*, Springer-Verlag, Berlin, 1992

[2] A. Dimca, *Hyperplane arrangements. An introduction*, Springer, Cham, 2017.

[3] R. Randell, *The fundamental group of the complement of a union of complex hyperplanes*, Invent. Math. 69 (1982), no. 1, 103–108

[4] M. Salvetti, *Topology of the complement of real hyperplanes in* $\mathbb{C}^n$, Invent. Math. 88 (1987), no. 3, 603–618.

[5] W. Arvola, *The fundamental group of the complement of an arrangement of complex hyperplanes*, Topology 31 (1992), no. 4, 757–765

[6] A. Hattori, *Topology of* C^n *minus a finite number of affine hyperplanes in general position*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 22 (1975), no. 2, 205–219.

[7] P. Deligne, *Les immeubles des groupes de tresses généralisés*, Invent. Math. 17 (1972), 273–302.

19. Yau's Solution of the Calabi Conjecture .... Dr A. G. Kovalev

The subject area of this essay is compact Kähler manifolds. Very informally, a Kähler manifold is a complex manifold admitting a metric and a symplectic form, both nicely compatible with the complex structure. The Ricci curvature of a Kähler manifold may be equivalently expressed as a differential form which is necessarily closed. Furthermore, the cohomology class defined by this form depends only on the complex manifold, but not on the choice of Kähler metric. The Calabi conjecture determines which differential forms on a compact complex manifold can be realized by Ricci forms of some Kähler metric. Substantial progress on the conjecture was made by Aubin and it was eventually proved by Yau. This result gives, among other things, a powerful way to find many examples of Ricci-flat manifolds. The essay could discuss aspects of the proof and possibly consider some applications and examples. Interested candidates are welcome to contact `A.G.Kovalev@dpmms` for further details.

Relevant Courses

Essential: Differential Geometry, Complex Manifolds

Useful: Algebraic Topology, Elliptic Partial Differential Equations

References

[1] D. Joyce, *Riemannian holonomy groups and calibrated geometry*, OUP 2007. Chapters 6 and 7.

[2] S.-T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation. I. *Comm. Pure Appl. Math.*, **31** (1978), 339–411.

[3] a good text on Kähler complex manifolds, e.g. D. Huybrechts, *Complex geometry. An introduction.* Springer 2005.

20. Dirac Operators .... Dr A. G. Kovalev

The Dirac operator, for smooth functions from $\mathbf{R}^n$ to $\mathbf{C}^N$, may be defined as a first order differential operator whose square is the Laplacian. (Thus the simplest example of Dirac operator would be the usual derivative of complex-valued functions on $\mathbf{R}$.) Unlike the Laplacian, which is well-defined on every oriented Riemannian manifold, the construction of Dirac operator requires the existence of a certain vector bundle, called the spinor bundle, over the base manifold. The essay could begin by explaining the significance of spinor bundles (cf. [1]), and why a Dirac operator can always be constructed when the dimension of the base manifold is 3 or 4. The kernel of a Dirac operator arises in many geometric and topological applications, including the Riemannian holonomy, deformations of volume-minimizing submanifolds, invariants of smooth 4-dimensional manifolds. The essay has an option to consider some of these topics. Interested candidates are welcome to contact `A.G.Kovalev@dpmms` and discuss the possibilities. The first two or three sections in [2] would be a good introductory reading (and a source of useful exercises!).

Relevant Courses

Essential: Differential Geometry, Algebraic Topology

Useful: Complex Manifolds

References

[1] J. Roe, *Elliptic operators, topology and asymptotic methods*, Pitman Res. Notes in Math., Longman, 1988 (or second edition 1998).

[2] N. Hitchin, *The Dirac operator.* Invitations to geometry and topology, Oxford Univ. Press, Oxford, 2002, pages 208–232.

[3] N. Hitchin, *Harmonic spinors*, Advances in Math. **14** (1974), 1–55.

[4] H.B. Lawson, Jr. and M.-L. Michelson, *Spin geometry*, Princeton University Press, 1989.

21. (No) Wandering Domains .... Dr H. Krieger

A holomorphic self-map f of the Riemann sphere can have stable regions - known as *Fatou components* - where the long-term behaviour of points under iteration is predictable. Sullivan's celebrated No Wandering Domains theorem [5] establishes that these components do not wander: that is, if U is a Fatou component of f, then the set $\{U, f(U), f^{\circ 2}(U), f^{\circ 3}(U), \dots\}$ is a finite collection of components.

In this essay, you will learn the basic theory of complex dynamics in one variable [4], and apply it to understand the proof of Sullivan's theorem. You can then proceed in a number of directions: (1) wandering domains in transcendental dynamics [2], (2) wandering domains in higher-dimensional complex dynamics [1], or (3) no wandering domains in p-adic dynamics [3]. In each case, you will first develop the basic dynamical theory for the relevant setting.

Relevant Courses

Essential: Part II Riemann Surfaces, Differential Geometry (Part II or Part III).

Useful: Algebraic Geometry (for direction (2)), Number Fields / Theory (for direction (3)).

References

[1] M. Astorg, X. Buff, R. Dujardin, H. Peters, J. Raissy. A two-dimensional polynomial mapping with a wandering Fatou component, *Annals of Mathematics* (2016), **184**, 263–313.

[2] I. N. Baker. Wandering domains in the iteration of entire functions, *Proc. London Math. Soc.* (1984) **49**, no. 3, 563–576.

[3] R. Benedetto. p-adic dynamics and Sullivan's no wandering domains theorem, *Compositio Mathematica* (2000), **122**, 281–298.

[4] L. Carleson, T. Gamelin. Complex dynamics. Springer Universitext Tracts in Mathematics (1993), Springer-Verlag New York, Inc.

[5] D. Sullivan. Quasiconformal homeomorphisms and dynamics I: Solution of the Fatou-Julia problem on wandering domains, *Annals of Mathematics* (1985), **122** no. 2, 401–418.

22. Effective Diophantine Approximation and Unlikely Intersections .... Dr H. Krieger

The complex plane $\mathbb{C}$ parametrizes isomorphism classes of elliptic curves via the j-invariant. The principle of unlikely intersections predicts that a curve $f(x, y) = 0$ in $\mathbb{C}^2$ with no modular component should contain only finitely many points for which both coordinates are the j-invariant of an elliptic curve which admits an additional structure (known as complex multiplication). This finiteness was established by André in 1998, but his proof was ineffective; that is, it did not provide for a given curve any way to find all points on the curve with this property. In 2012 an effective version was proved independently by Kühne [5] and Bilu-Masser-Zannier [2], using the theory of linear forms in logarithms. This is a special case of what is known as the *effective André-Oort conjecture.*

The main goal of this essay will be to understand the theory of Weil heights in arithmetic geometry (see [4]) and the technique of linear forms in logarithms (see [3]), and to explain how they are used to provide effective bounds for questions of unlikely intersections as discussed above. An interested student might then proceed to related questions of the arithmetic geometry of the complex plane as moduli space of elliptic curves such as [1], or other instances of unlikely intersections (see [6]), or further results in effective Diophantine geometry (see [3]).

Relevant Courses

Essential: Part II Number Fields.

Useful: Part III Algebraic Number Theory, Elliptic Curves, and Algebraic Geometry.

References

[1] Y. Bilu, P. Habegger, L. Kühne. Effective bounds for singular units. Preprint arXiv:1805.07167.

[2] Y. Bilu, D. Masser, U. Zannier. An effective Theorem of André for CM-points on a plane curve, *Math. Proc. Camb. Phil. Soc.* (2013), **154**, 145–152.

[3] Y. Bugeaud. Linear Forms in Logarithms and Applications, IRMA Lectures in Mathematics and Theoretical Physics, 28, Zürich: European Mathematical Society, 2018.

[4] M. Hindry, J. H. Silverman. Diophantine Geometry: An Introduction. Springer Graduate Texts in Mathematics (2000), Springer-Verlag New York, Inc.

[5] L. Kühne. An effective result of André-Oort type, *Annals of Mathematics* (2012), **176**, 651–671.

[6] U. Zannier. Some problems of unlikely intersections in arithmetic and geometry. Annals of mathematics studies (2012), Princeton University Press.

23. Independence Results for Basic Axioms of Set Theory .... Dr B. Löwe

The basic axiom systems of set theory are combined from the axioms (and axiom schemes) of Extensionality (Ext), Pairing (Pair), Union (Un), Power Set (Pow), Separation (Sep), Infinity (Inf), Replacement (Repl), and Regularity (Reg). Finite Set Theory (FST) is $\mathsf{Ext} + \mathsf{Pair} + \mathsf{Un} + \mathsf{Pow} + \mathsf{Sep}$, Zermelo Set Theory ($\mathsf{Z}$) is $\mathsf{FST} + \mathsf{Inf}$, Zermelo-Fraenkel Set Theory without Foundation (ZF_0) is $\mathsf{Z} + \mathsf{Repl}$, and Zermelo-Fraenkel Set Theory (ZF) is $\mathsf{ZF}_0 + \mathsf{Reg}$.

As a consequence, there are $2^5 = 32$, $2^6 = 64$, $2^7 = 128$, or $2^8 = 256$ subsystems of FST, Z, ZF_0, and ZF given by these axioms, respectively. The standard literature usually discusses only very few cases, usually just that

$$\mathsf{FST} < \mathsf{Z} < \mathsf{ZF}_0 < \mathsf{ZF},$$

where $S < T$ means that T implies S, but there is a structure satisfying S, but not T [1, Chapter IV].

The goal of this essay is to explore the lattice of these subsystems in terms of finding structures that show that two of the subsystems are not logically equivalent.

As a first step, one would consider the subsystems of FST. When moving to subsystems of Z, one needs to discuss the Axiom of Infinity in more detail (since in its usual formulation, it needs a number of other axioms to hold).

If there is time for further exploration, one could consider the work by Mathias on weak systems of set theory, in particular, models of Z [2].

Relevant Courses

Essential: Part II Logic and Set Theory (or equivalent).

Useful: Part III Topics in Set Theory (or equivalent).

References

[1] Kenneth Kunen. *Set theory. An introduction to independence proofs.* Studies in Logic and the Foundations of Mathematics, Vol. 102. (North-Holland, 1980).

[2] Adrian R. D. Mathias. Slim models of Zermelo Set Theory. *Journal of Symbolic Logic* 66 (2001):487–496.

24. The Generalised Real Numbers .... Dr B. Löwe

Generalised analysis is dealing with analytical, metric, and topological properties of generalisations of the set of real numbers to uncountable cardinals κ. In recent years, there has been a large number of new results in this field and a list of open problems was published in 2016 [4].

A fundamental problem in this field was the search for the correct analogue of the real number line for uncountable cardinals κ. In [3], Lorenzo Galeotti provided an answer to this question by defining the *κ-reals* based on Conway's surreal numbers [2].

This essay aims at developing the basic theory of Galeotti's κ-reals and studying some of their properties that distinguish them from the classical real numbers as well as other candidates for the generalised reals (such as Sikorski's *long reals*). A useful reference is [1].

Relevant Courses

Essential: Part IB Metric and Topological Spaces (or equivalent), Part II Logic and Set Theory (or equivalent)

Useful: Part III Topics in Set Theory (or equivalent).

References

[1] Merlin Carl, Lorenzo Galeotti, Benedikt Löwe. The Bolzano-Weierstra theorem in generalised analysis. To appear in: *Houston Journal of Mathematics* 44 (2018).

[2] John H. Conway. *On Numbers and Games.* (A K Peters & CRC Press, 2000).

[3] Lorenzo Galeotti, A candidate for the generalised real line. In: *Pursuit of the Universal, 12th Conference on Computability in Europe, CiE 2016, Paris, France, June 27–July 1, 2016, Proceedings.* Edited by A. Beckmann, L. Bienvenu, and N. Jonoska. (Springer-Verlag, 2016), 271–281.

[4] Yurii Khomskii, Giorgio Laguzzi, Benedikt Löwe, Ilya Sharankou. Questions on Generalised Baire Spaces. *Mathematical Logic Quarterly* 62:4-5 (2016), 439–456.

25. Long Blackwell Games .... Dr B. Löwe

In the theory of infinite games, there is usually a trade-off between the (transfinite) length of the game and the size of the possible set of moves. E.g., if $\mathsf{AD}_X[\alpha]$ denotes the axiom of determinacy for games of length α with moves in X, then Kechris proved that $\mathsf{AD}_{\mathbb{N}}[\omega^2]$ is equivalent to $\mathsf{AD}_{\mathbb{R}}[\omega]$.

Blackwell's *infinite games with slightly imperfect information* [1] are not easily extended to transfinite length. Based on an idea by de Kloet [2, Section 2.4], transfinite Blackwell games for some ordinals were defined in [3, Section 9.2].

The goal of this essay is to develop the basic theory of long Blackwell games along the definitions from [3, Section 9.2], study Kechris's theorem connecting the length of games and the size of the set of possible moves, and link them.

Relevant Courses

Essential: Part II Logic and Set Theory (or equivalent), Part II Probability and Measure (or equivalent)

Useful: Part III Topics in Set Theory (or equivalent).

References

[1] David Blackwell, Games with Infinitely Many Moves and Slightly Imperfect Information. In: *Games of no chance, Combinatorial games at MSRI, Workshop, July 11–21, 1994 in Berkeley, CA, USA, Cambridge 1997.* Edited by R. J. Nowakowski. Mathematical Sciences Research Institute Publications, Vol. 29. (Cambridge University Press, 1996), 407–408.

[2] Daisuke Ikegami, David de Kloet, Benedikt Löwe, The Axiom of Real Blackwell Determinacy, *Archive for Mathematical Logic* 51 (2012), 671–685.

[3] Benedikt Löwe, Set Theory of Infinite Imperfect Information Games. In: *Set Theory: Recent Trends and Applications.* Edited by A. Andretta. Quaderni di Matematica, Vol. 17. (Arachne, 2005), 137–181.

26. Group Cohomology from the Topological Viewpoint .... Dr O. Randal-Williams

Group cohomology attaches to each group G and commutative ring k a graded-commutative k-algebra $H^*(G;k)$, which one may view as an attempt to "linearise" the group G. This may be defined and studied purely algebraically, but it is extremely profitable to instead consider it as the singular cohomology of a certain topological space BG associated to the group G, so that one may use techniques from algebraic topology to study it: this point of view reveals it as a special case of so-called equivariant cohomology.

In this essay you should first give an introduction to this subject, discussing basic results such as the finite-generation of cohomology of compact Lie groups and the Localisation Theorem for equivariant cohomology. You should then explain Quillen's theorem [2] relating the ring-theory of $H^*(G;\mathbb{F}_p)$ (its Krull dimension) to the group-theory of G (the rank of its maximal elementary abelian p-subgroup), and generalisations. Finally, you should explain Symonds' recent proof [3] of Benson's Regularity Conjecture, which establishes a strong ring-theoretic property (Castelnuovo–Mumford regularity equals zero) for the cohomology of any finite group.

Along the way you will need to learn some pieces of Algebraic Topology, such as spectral sequences and characteristic classes.

Relevant Courses

Essential: Algebraic Topology

Useful: Algebras, or a willingness to pick up a certain amount of commutative algebra as you go. Algebraic Geometry, or a willingness to learn a bit about sheaves.

References

[1] D. Benson, *Representations and cohomology. II. Cohomology of groups and modules.* Cambridge Studies in Advanced Mathematics, 31. Cambridge University Press, 1991.

[2] D. Quillen, *The spectrum of an equivariant cohomology ring. I.* Ann. of Math. (2) 94 (1971), 549-572

[3] P. Symonds, *On the Castelnuovo–Mumford regularity of the cohomology ring of a group.* J. Amer. Math. Soc. 23 (2010), no. 4, 1159-1173.

[4] B. Totaro, *Group cohomology and algebraic cycles.* Cambridge Tracts in Mathematics, 204. Cambridge University Press, 2014.

27. Surface Bundles ....

Dr O. Randal-Williams

A surface bundle $\pi : E \to B$ is the analogue of a vector bundle in which vector spaces are replaced by surfaces. The study of surface bundles arises—directly or indirectly—in topology, group theory, differential and algebraic geometry, and arithmetic, and this multitude of perspectives makes it an especially interesting subject. This essay will focus on the algebraic topology of surface bundles, and especially on the characteristic classes κ_i of surface bundles introduced by Miller, Morita, and Mumford.

After explaining how to construct the κ_i, you should show that they are not zero and indeed are algebraically independent in rational cohomology as the genus of the surface tends to infinity. This might be done following Miller's generalisation [2] of a construction of Atiyah, or else by a surprising argument of Akita–Kawazumi–Uemura [1] using cyclic group actions on surfaces.

There are many directions to go after this, which can be discussed with me. One choice would be to explain the existence of algebraic relations among the κ_i when one does not pass to the infinite genus limit, perhaps starting from Morita's relations [3] constructed using the Abel–Jacobi map.

Relevant Courses

Essential: Part III Algebraic Topology, Part II Algebraic Geometry and Part II Riemann Surfaces.

References

[1] T. Akita, N. Kawazumi, T. Uemura, *Periodic surface automorphisms and algebraic independence of Morita-Mumford classes.* J. Pure Appl. Algebra 160 (2001), no. 1, 1-11.

[2] E. Y. Miller, *The homology of the mapping class group.* J. Differential Geom. 24 (1986), no. 1, 1-14.

[3] S. Morita, Shigeyuki *Families of Jacobian manifolds and characteristic classes of surface bundles. I.* Ann. Inst. Fourier 39 (1989), no. 3, 777-810.

[4] S. Morita, *Geometry of characteristic classes.* Translated from the 1999 Japanese original. Translations of Mathematical Monographs, 199. Iwanami Series in Modern Mathematics. American Mathematical Society, Providence, RI, 2001.

28. E_k-algebras ....
Dr O. Randal-Williams

In homotopy theory many *equations* from classical algebra (e.g. being commutative: $a \cdot b = b \cdot a$) must be replaced by *homotopies* (e.g. a homotopy from $(a, b) \mapsto a \cdot b$ to $(a, b) \mapsto b \cdot a$). This additional data might also be subject to constraints, but in the same spirit these should not be in the form of equations, but rather as further homotopies. One may decide how far along this process to go, giving a hierarchy of structures

$$E_1 \supset E_2 \supset E_3 \supset \cdots \supset E_\infty$$

interpolating between the homotopical analogue of associative (E_1) and of fully commutative (E_∞). While these extreme notions are the most common, the intermediate types of commutativity, especially E_2, play important roles.

This essay will explore the classical theory of E_k-algebras [4] and their relation to k-fold loop spaces, homology operations for E_k-algebras [1], and then give an exposition of some recent results [2, 3] applying E_k-algebras to study homological stability. It is quite open-ended and you should discuss your plans *in detail* with me first.

Relevant Courses

Essential: Part III Algebraic Topology.

References

[1] F. R. Cohen, T. J. Lada, J. P. May, *The homology of iterated loop spaces.* Lecture Notes in Mathematics, Vol. 533, 1976.

[2] S. Galatius, A. Kupers, O. Randal-Williams *Cellular E_k-algebras.* arXiv:1805.07184.

[3] S. Galatius, A. Kupers, O. Randal-Williams *E_2-cells and mapping class groups.* arXiv:1805.07187.

[4] J. P. May, *The geometry of iterated loop spaces.* Lectures Notes in Mathematics, Vol. 271, 1972.

29. The Heegaard Floer Contact Invariant ....
Professor J. A. Rasmussen

A contact structure ξ on a 3-manifold Y is a 2-dimensional sub-bundle of the tangent bundle which is maximally nonintegrable, in the sense that if $\xi = \ker d\alpha$ for some $\alpha \in \omega^1(Y)$, then $\alpha \wedge d\alpha \neq 0$. Contact structures arise naturally as the boundary of symplectic four-manifolds. Heegaard Floer homology is a package of 3-manifold invariants defined by Ozsváth and Szabó. One part of this package is an invariant of contact 3-manifolds $c(\xi)$.

The essay should define this invariant and describe some of its properties, including the fact that it is nonvanishing for fillable contact structures and 0 for overtwisted structures. This will require some discussion of Lefshetz fibrations and the Giroux correspondence. The basic properties of Heegaard Floer homology can be treated as a black box. Time and space permitting, you may want to discuss an application/extension of the invariant. Possible choices include the construction of tight contact structures on Seifert fibred spaces, the construction of a tight contact structure with $c(\xi) = 0$, or the extension of the contact invariant to sutured Floer homology.

Relevant Courses

Essential: Algebraic Topology, Differential Geometry

Useful: 3-Manifolds, Symplectic Topology

References

[1] K. Honda. W. Kazez, and G. Matic, On the contact class in Heegaard Floer homology, *JDG* 83 (2009), 289-311. arXiv:math/0609734.

[2] B. Ozbagci and A. Stipsicz, *Surgery on Contact Manifolds and Stein Surfaces*, Springer, 2004.

[3] P. Ozsváth and Z. Szabó, An introduction to Heegaard Floer homology, in *Floer Homology, Gauge Theory, and Low-Dimensional Topology*, edited by D. Ellwood et al. Clay Mathematics Institute, 2006.

[4] P. Ozsváth and Z. Szabó, Heegaard Floer homology and contact structures, *Duke Math. J.* 129 (2005), 39-61. arXiv:math/0210127.

30. Annular Khovanov Homology .... Professor J. A. Rasmussen

Khovanov homology is an invariant of links in R^3. It admits a more or less elementary combinatorial description, but has ties to many rich and interesting geometric theories. Annular Khovanov homology is a related invariant of links in the solid torus $S^1 \times D^2$. The main goal of this essay is to understand a theorem of Grigsby, Licata, and Wehrli which says that annular Khovanov homology of a link L is naturally a representation of the Lie algebra $\mathfrak{sl}(2)$.

The essay should explain the definition of the annular Khovanov homology and discuss Grigsby Licata and Wehrli's construction in detail. It should then go on to discuss a subject of your choice related to annular Khovanov homology. Possible topics include the relationship between annular Khovanov homology and sutured Floer homology (due to Grigsby and Wehrli), Khovanov's categorification of the Burau representation, or Queffelec and Rose's construction of the annular $\mathfrak{sl}(n)$ homology and the $\mathfrak{sl}(n)$ action on it.

Relevant Courses

Essential: None

Useful: Lie Algebras and their Representations, Categorified Knot Invariants

References

[1] J.E. Grigsby, A. Licata, and S. Wehrli, Annular Khovanov homology and knotted Schur-Weyl representations, *Compos. Math.* 154 (2017), 459-502. arXiv: 1505:04386.

[2] J.E. Grigsby and S. Wehrli, Khovanov homology, sutured Floer homology and annular links, Alg. Geom. Top. 10 (2010) 2009-2039. arXiv:0907.4375

[3] H. Queffelec and D. Rose, Sutured annular Khovanov-Rozansky homology, *Trans AMS* 370 (2018) 1285-1319, arXiv:1506.08188.

31. p-adic Modular Forms ....

Dr G. Rosso

The main objective of the essay will be to study level 1 modular forms and congruences modulo p among them; in particular you shall prove a structure theorem for modular forms modulo p, and how this can be used to define modular forms with p-adic weights.
If times allows, the problem of when a p-adic modular form is classical could be addressed.

Relevant Courses

Essential: Algebraic Number Theory, Elliptic Curves

Useful: Algebraic Geometry

References

[1] Serre, J.-P., A course in arithmetic, Chapter VII. Graduate Texts in Mathematics, No. 7. Springer-Verlag, New York-Heidelberg, 1973. viii+115 pp.

[2] Swinnerton-Dyer, H. P. F., On l-adic representations and congruences for coefficients of modular forms. Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, 1972), pp. 155. Lecture Notes in Math., Vol. 350, Springer, Berlin, 1973

[3] Serre, Jean-Pierre, Formes modulaires et fonctions zéta p-adiques. (French) Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, 1972), pp. 191268. Lecture Notes in Math., Vol. 350

[4] Katz, Nicholas M., p-adic properties of modular schemes and modular forms. Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), pp. 69190. Lecture Notes in Mathematics, Vol. 350, Springer, Berlin, 1973.

[5] Coleman, Robert F., Classical and overconvergent modular forms. Invent. Math. 124 (1996), no. 1-3, 215241.

32. Symplectic Structures on Euclidean Space ....

Professor I. Smith

There are various constructions of non-standard "exotic" symplectic structures on Euclidean space; some are constructed by hand, but the most interesting arise from contractible affine varieties. Their exotic nature is often tied up with the existence of interesting Lagrangian submanifolds and the behaviour of dynamical systems on the manifold, as probed by holomorphic curve counting invariants (Floer theory, symplectic cohomology); there are many open questions about how much of the underlying affine algebraic geometry is captured by symplectic topology. This essay will construct some examples, prove they are exotic (taking some input from holomorphic curve theory as a black box where necessary), and discuss open questions.

Relevant Courses

Essential: Algebraic Topology, Differential Geometry, Symplectic Topology

Useful: Algebraic Geometry, Complex Manifolds, Topics in Floer theory

References

[1] Paul Seidel and Ivan Smith. *The symplectic topology of Ramanujam's surface.* Comment. Math. Helv. 80 (2005), arXiv:math/0411601.

[2] Paul Seidel. *Simple examples of distinct Liouville type symplectic structures.* J. Topol. Anal. 3 (2011), arXiv:1011.1415

[3] Michele Audin, Francois Lalonde, Leonid Polterovich. *Symplectic rigidity: Lagrangian submanifolds.* Chapter X in "Holomorphic curves in symplectic geometry". Birkhäuser, 1994.

[4] Mark McLean. *Symplectic invariance of uniruled affine varieties and log Kodaira dimension.* Duke Math J. 163 (2014), arXiv:1211.2034

33. Packing Symplectic Tori .... Professor I. Smith

When can one fill the volume of a symplectic manifold by a collection of disjointly embedded standard symplectic balls of equal radius? (Or even one such ball?) Obstructions to such "packings" which go beyond volume constraints are bound up with many fundamental aspects of symplectic topology. One can relate configurations of embedded symplectic balls with symplectic forms on blow-ups of the given manifold, so symplectic packings are closely connected to finding the cone of cohomology classes containing symplectic forms on a blow-up. Recent work has led to a complete solution of the packing problem on symplectic tori with linear symplectic forms; the (ir)rationality of the form in cohomology plays a key role. This essay will outline the proof in the four-dimensional case, and perhaps say something in higher dimensions.

Relevant Courses

Essential: Algebraic Topology, Differential Geometry, Symplectic Topology, Complex Manifolds

Useful: Algebraic Geometry

References

[1] Janko Latschev, Dusa McDuff and Felix Schlenk. *The Gromov width of 4-dimensional tori.* Geom. Topol. 17 (2013), arXiv:1111.6566.

[2] Michael Entov and Misha Verbitsky. *Unobstructed symplectic packing for tori and hyperkahler manifolds.* J. Topol. Anal. 8 (2016), arXiv:1412.7183

[3] Paul Biran. *Symplectic packing in dimension four.* Geom. Funct. Anal. 7 (1997), arXiv:math/9606001

34. Dynamics on K3 Surfaces .... Professor I. Smith

K3 surfaces are all diffeomorphic to a smooth quartic surface in complex projective 3-space. They play a special role in the classification of complex surfaces, and have rich complex dynamics. The entropy of a holomorphic automorphism of a complex algebraic variety is given by the logarithm of its spectral radius for the action on cohomology, which means that dynamical questions can be approached lattice-theoretically. K3 surfaces admit automorphisms of positive entropy given by Salem numbers. Their construction makes extensive use of the Torelli theorem for K3 surfaces, Coxeter groups, and more. This essay will explain the Torelli theorem, discuss the Gromov-Yomdin theorem on topological entropy, construct some explicit positive entropy automorphisms, and discuss open questions.

Relevant Courses

Essential: Algebraic Topology, Differential Geometry, Complex Manifolds

Useful: Algebraic Geometry.

References

[1] Curt McMullen. *Dynamics on K3 surfaces: Salem numbers and Siegel discs.* J. Reine Angew. Math. 545 (2002).

[2] Curt McMullen. *Automorphisms of projective K3 surfaces with minimum entropy.* Invent. Math. 203 (2016).

[3] Mikhail Gromov. *On the entropy of holomorphic maps.* Enseign. Math. 49 (2003).

[4] W. Barth, C. Peters, A. Van de Ven. *Compact complex surfaces.* Ergeb. der Mathematik, Springer 1994.

35. Expansion and Robust Expansion .... Professor A. G. Thomason

A graph is an expander if every set of vertices has a substantial sized neighbourhood; a typical definition might be that $|\Gamma(S)| \geq \lambda|S|$ for every set $S \subset V(G)$ with $|S| \leq |V(G)|/2$. Expansion has long been recognised to be a very useful property of graphs, both theoretically and in applications; it is easy to construct, say, hamiltonian cycles or large minors (subcontractions) in expanders, and expanding networks offer fast communication. More recently, a notion of *robust* expansion has emerged, allowing stronger constructions and leading to the proofs of not a few outstanding conjectures, such as Kelly's, that every regular tournament decomposes into hamiltonian cycles. An essay could outline the consequences of expansion and how robust expansion differs, giving some examples of each in detail.

Relevant Courses

Essential: None

Useful: Combinatorics

References

[1] S. Hoory, N. Linial and A. Widgerson, Expander graphs and their applications, *Bull. Amer. Math. Soc.* **43** (2006), 439–561.

[2] M. Krivelevich and B. Sudakov, Minors in expanding graphs, *Geom. Funct. Anal.* **19** (2009), 294–331.

[3] D. Kühn, A. Lo, D. Osthus and K. Staden, The robust component structure of dense regular graphs and applications, *Proc. London Math. Soc.* **110** (2015), 19–56.

36. Sidorenko's Conjecture .... Professor A. G. Thomason

Let H be some fixed graph. What is the minimum number of copies of H that appear in a large graph G of density p? Even if H is K_3 this is a very difficult question, which was answered only recently. But, in the case that H is biparitite, Sidorenko made the remarkable conjecture that the minimum is achieved by a random (or random-like) graph G. The conjecture has been proved for some H; curiously, it's known for sparse H such as trees, and for dense H, such as complete bipartite, but not for H in between. An essay would look at the conjecture, some elementary arguments, and more recent arguments using entropy or dependent random choice.

Relevant Courses

Essential: Combinatorics

References

[1] D. Conlon, J.H. Kim C. Lee and J. Lee, Some advances on Sidorenko's conjecture, *J. London Math Soc* (2018), online.

[2] A. Sidorenko, A correlation inequality for bipartite graphs, *Graphs and Combinatorics* **9** (1993), 201–204.

[3] B. Szegedy, An information theoretic approach to Sidorenko's conjecture, *arXiv*:1406.6738.

37. Cohomology of Number Fields .... Professor J. A. Thorne

Group cohomology assigns to any group G and $\mathbb{Z}[G]$-module M a series of abelian groups $H^i(G, M)$. When $G = \mathrm{Gal}(L/K)$ is the Galois group of a field extension and M is a module of arithmetic interest (for example, $M = L^\times$), these groups have arithmetic meaning, and are commonly referred to as Galois cohomology groups. When K is a number field and G is the absolute Galois group of K, determination of the Galois cohomology of the units and of the idèle class group is essentially equivalent to class field theory.

The goal of this essay will be to prove the main theorems of global class field theory using Galois cohomology. A good essay will go further. One possible direction would be to discuss the Poitou–Tate duality theorems in Galois cohomology. These duality theorems are analogous to Poincaré duality for manifolds in algebraic topology. Another possible direction would be to discuss the application of global class field theory to the reciprocity law for the power residue symbol, which generalises quadratic reciprocity.

Relevant Courses

Essential: Algebraic Number Theory

Useful: Elliptic Curves

References

[1] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg, *Cohomology of number fields.* Grundlehren der Mathematischen Wissenschaften, 323. Springer-Verlag, Berlin, 2008.

[2] Emil Artin and John Tate, *Class field theory.* AMS Chelsea Publishing, Providence, RI, 2009.

[3] John Tate, *Global class field theory.* in *Algebraic Number Theory* (Proc. Instructional Conf., Brighton, 1965), pp. 162–203, Thompson, Washington, D.C., 1967.

38. Arithmetic Statistics of Elliptic Curves .... Professor J. A. Thorne

Let p be a prime. If E is an elliptic curve over $\mathbb{Q}$, it has an associated p-Selmer group $\mathrm{Sel}_p(E)$. This is an $\mathbb{F}_p$-vector space whose dimension gives an upper bound for the rank of E (i.e. the dimension of $E(\mathbb{Q}) \otimes \mathbb{Q}$). In recent years many mathematicians have studied the following question: what is the distribution of the quantity $d_{p,E} = \dim_{\mathbb{F}_p} \mathrm{Sel}_p(E)$ as the curve E varies? What is the average value? What about other statistics, e.g. the average size of the p-Selmer group?

The goal of this essay will be to explore some of the heuristics and theorems that have recently appeared about this question. A good essay should aim to discuss both heuristics and theorems, perhaps starting with the work of Poonen–Rains (that formulates precise heuristics, that are amenable to generalization) and proceeding to the work of Swinnerton-Dyer (which proves a result about the distribution of $d_{2,E}$ in certain quadratic twist families). An alternative reference is the paper by Klagsbrun, Mazur, and Rubin, which proves similar results for an elliptic curve over an arbitrary number field. Both references use Markov chains to describe the variation of the Selmer group under quadratic twists.

Relevant Courses

Essential: Elliptic Curves

Useful: Algebraic Number Theory

References

[1] Bjorn Poonen and Eric Rains, *Random maximal isotropic subspaces and Selmer groups.* J. Amer. Math. Soc. 25 (2012), no. 1, pp. 245–269.

[2] Manjul Bhargava, Daniel Kane, Hendrik Lenstra, Bjorn Poonen, and Eric Rains, *Modeling the distribution of ranks, Selmer groups, and Shafarevich–Tate groups of elliptic curves.* Camb. J. Math. 3 (2015), no. 3, pp. 275–321.

[3] Peter Swinnerton-Dyer, *The effect of twisting on the 2-Selmer group.* Math. Proc. Cambridge Philos. Soc. 145 (2008), no. 3, pp. 513–526.

[4] Zev Klagsbrun, Barry Mazur, and Karl Rubin, *A Markov model for Selmer ranks in families of twists.* Compos. Math. 150 (2014), no. 7, pp. 1077–1106.

39. Komlós's Conjecture in Discrepancy Theory ....

Dr P. P. Varjú

Let n and d be two integers and let $u_1, \ldots, u_n \in \mathbf{R}^d$ be a sequence of vectors with $\|u_j\|_2 \leq 1$ for all $j = 1, \ldots, n$. A conjecture of Komlós predicts the existence of an absolute constant C (it is independent of both n and d !) such that there are numbers $\varepsilon_j \in \{-1, 1\}$ with

$$\|\varepsilon_1 u_1 + \ldots + \varepsilon_n u_n\|_\infty \leq C.$$

To see where this is coming form, consider the standard basis of $\mathbf{R}^d$ in the role of u_j. The essay will discuss progress towards this remarkable conjecture.

Relevant Courses

No courses are required but basic knowledge of Probability is very useful for this essay.

References

[1] Banaszczyk, W. *Balancing vectors and convex bodies.* Studia Math. 106 (1993), no. 1, 93–100.

[2] Banaszczyk, W. *Balancing vectors and Gaussian measures of n-dimensional convex bodies.* Random Structures Algorithms 12 (1998), no. 4, 351–360.

[3] Bansal, N. ; Dadush D. ; Garg S. ; Lovett S. *The Gram-Schmidt Walk: A Cure for the Banaszczyk Blues.* 2017. https://arxiv.org/abs/1708.01079v1

[4] Beck, J. ; Fiala, T. *"Integer-making" theorems.* Discrete Appl. Math. 3 (1981), no. 1, 1–8.

[5] Bukh, B. *An improvement of the Beck-Fiala theorem.* Combin. Probab. Comput. 25 (2016), no. 3, 380–398.

[6] Spencer, J. *Six standard deviations suffice.* Trans. Amer. Math. Soc. 289 (1985), no. 2, 679–706.

40. Equidistribution of Roots of Polynomials ....

Dr. P. P. Varjú

The essay will discuss the phenomenon that polynomials with small integer coefficients tend to have their roots accumulated near the unit circle and they are approximately evenly distributed there. Precise statements of this kind have been proved by Erdős and Turán [3] and Bilu [2]. These results have been revisited by many authors because of their importance to number theory.

Relevant Courses

No courses are required but basic knowledge of Galois theory and Fourier analysis is very useful for this essay.

References

[1] Amoroso, F. ; Mignotte, M. *On the distribution of the roots of polynomials.* Ann. Inst. Fourier (Grenoble) 46 (1996), no. 5, 1275–1291.

[2] Bilu, Y. *Limit distribution of small points on algebraic tori.* Duke Math. J. 89 (1997), no. 3, 465–476.

[3] Erdős, P. ; Turán, P. On the distribution of roots of polynomials. Ann. of Math. (2) 51, (1950). 105–119.

[4] Granville, A. *The distribution of roots of a polynomial.* Equidistribution in number theory, an introduction, 93–102, NATO Sci. Ser. II Math. Phys. Chem., 237, Springer, Dordrecht, 2007.

[5] Petsche, C. *A quantitative version of Bilu's equidistribution theorem.* Int. J. Number Theory 1 (2005), no. 2, 281–291.

[6] Rumely, R. *On Bilu's equidistribution theorem.* Spectral problems in geometry and arithmetic (Iowa City, IA, 1997), 159–166, Contemp. Math., 237, Amer. Math. Soc., Providence, RI, 1999.

[7] Soundararajan, K. *Equidistribution of zeros of polynomials.* 2018. https://arxiv.org/abs/1802.06506

41. Croot-Sisask Almost-Periodicity and Applications ....

Dr J. Wolf

Originally developed for the purpose of strengthening results on the existence of long arithmetic progressions in sum sets of subsets of the integers, the Croot-Sisask almost-periodicity technique has been instrumental in several recent breakthroughs: Sanders employed it to prove almost-logarithmic bounds in Roth's theorem, and Schoen and Sisask used it to obtain essentially tight bounds in Roth's theorem with four variables. It has thus become firmly established as an essential component of the toolkit of modern additive combinatorics.

The essay will motivate, describe and prove the original almost-periodicity result (there are by now several proofs available in the literature), and then go on to cover at least two of its more substantial applications. While the technique itself is purely combinatorial (or probabilistic, depending on one's point of view), familiarity with the discrete Fourier transform is indispensable for making sense of the applications. A good essay will devote considerable attention to developing the necessary background on regular Bohr sets as substructures that are approximately closed under addition.

Relevant Courses

Essential: Introduction to Discrete Analysis.

Useful: Introduction to approximate groups.

References

[1] Croot, Ernie, and Olof Sisask. *A Probabilistic Technique for Finding Almost-Periods of Convolutions.* Geometric and Functional Analysis 20, no. 6 (2010): 136796.

[2] Schoen, Tomasz, and Olof Sisask. *Roth's Theorem for Four Variables and Additive Structures in Sums of Sparse Sets.* Forum of Mathematics Sigma, no. 4 (2016): e5e28.

[3] Sanders, Tom. *On Roth's Theorem on Progressions.* Annals of Mathematics. Second Series 174, no. 1 (2011): 61936.

42. Concentration and Functional Inequalities and their Relation to Markov Processes ....

Dr S. Andres

Consider a Markov process $(X_t)_{t\geq 0}$ and assume that X is stationary and ergodic, which ensures 'convergence to equilibrium', that is the convergence in law of X_t to the stationary distribution as $t \to \infty$. However, the rate of convergence (which is of interest in many areas, for example, in non-equilibrium statistical mechanics or Markov chain Monte Carlo algorithms) is unknown in general.

It turns out that the convergence to equilibrium can be analysed by certain functional inequalities such as Poincaré inequalities, Sobolev inqualities or Logarithmic Sobolev inequalities, which lead to (precise) quantitative bounds on the convergence to equilibrium and other properties of the transition semigroup of X such as ultra- and hypercontractivity.

On the basis of the monograph [1] and the lecture notes [2], a successful essay will describe in detail the relation between those functional inequalities and semigroup properties, and will discuss applications by means of at least one example.

Relevant Courses

Essential: Markov Chains, Advanced Probability

Useful: Applied Probability, Linear Analysis or Functional Analysis

References

[1] D. Bakry, I. Gentil, M. Ledoux. Analysis and geometry of Markov diffusion operators. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 348. Springer, Cham, 2014.

[2] R. van Handel. Probability in High Dimension.
Lecture notes, available at `https://web.math.princeton.edu/~rvan/`

43. Piecewise Deterministic Markov Processes ....

Dr S. A. Bacallado

Piecewise deterministic Markov processes (PDMPs) evolve according to an ordinary differential equation for random lengths of time between stochastic jumps in the state space. They were introduced by Davis in 1984 [1] and have recently found use in Markov chain Monte Carlo (see [2,3] and references within). They have the desirable property that the deterministic pieces can travel long distances for a limited computational cost, avoiding the diffusive behaviour of many reversible Markov chain samplers.

A more applied essay could review the main aspects of the algorithms discussed in [2, 3] and illustrate their application in a simple statistical problem. A more theoretical essay could focus on what can be proven about the mixing time of a PDMP, for example, through the analysis in the recent manuscript by Andrieu et al. [4].

Relevant Courses

Essential: None

Useful: Bayesian Modelling and Computation, Advanced Probability

References

[1] Davis, M. Piecewise-deterministic Markov processes: A general class of non-diffusion stochastic models. Journal of the Royal Statistical Society: Series B, pp. 353-388, (1984).

[2] Bierkens, J. et al. Piecewise deterministic Markov processes for scalable Monte Carlo on restricted domains. Statistics & Probability Letters, 136, pp. 148-154, (2018).

[3] Fearnhead, P. et al. Piecewise deterministic Markov processes for continuous-time Monte Carlo. Statistical Science, 33.3, pp. 386-412, (2018).

[4] Andrieu, C., Durmus, A., Nüsken, N, and Roussel, J. Hypercoercivity of Piecewise Deterministic Markov Process-Monte Carlo. Manuscript available at: https://arxiv.org/abs/1808.08592.

44. Random Matrix Eigenvalue Statistics .... Dr R. Bauerschmidt

Consider the *Gaussian Unitary Ensemble* (GUE): a complex hermitian $N \times N$ matrix with independent complex Gaussian entries above the diagonal. (The entries below the diagonal are then determined by the constraint that the matrix is hermitian.) The distribution of the N real eigenvalues of the GUE can be computed explicitly. This distribution has interesting structure: it is a determinantal point process. This allows, for example, the limiting distribution of the gap between two neighbouring eigenvalues or the distribution of the largest eigenvalue to be computed explicitly. These distributions turn out to be non-Gaussian, yet ubiquitous — or universal. For example, the rescaled distribution of the gaps between two zeroes on the critical line of the Riemann zeta function is conjectured to converge to the same distribution as the eigenvalue gap distribution of the GUE.

The goal of this essay is to derive the joint eigenvalue distribution of the GUE, to derive its global limiting density (the Wigner semicircle law), and to derive the determinantal structure of the eigenvalue distribution and its limiting correlation kernel (the Sine kernel). Then either a derivation of the limiting eigenvalue gap distribution (the Gaudin distribution) as well as the distribution of the largest eigenvalue (the Tracy–Widom distribution) should be given, or an introduction to the universality problem for random matrices.

Relevant Courses

Essential: Advanced Probability, Stochastic Calculus and Applications

References

[1] L. Erdős and H.-T. Yau. *A dynamical approach to random matrix theory*, volume 28 of *Courant Lecture Notes in Mathematics*. Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2017.

[2] T. Tao. *Topics in random matrix theory*, volume 132 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.

[3] G.W. Anderson, A. Guionnet, and O. Zeitouni. *An introduction to random matrices*, volume 118 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010.

45. Loop Erased Random Walk and SLE_2 ....
Dr J. P. Miller

A random walk on the vertices of a graph is a particle which in each time step moves to a neighbor of its currently location with equal probability. The loop erasure of a random walk is obtained from by starting with a random walk and then chronologically erasing the loops made by the random walk. It was introduced as a toy model for the so-called self-avoiding walk (SAW) by Lawler. It was proved by Lawler-Schramm-Werner that if one considers loop erased random walk on $\mathbb{Z}^2$, then in the fine mesh limit it coverges to the Schramm-Loewner evolution (SLE) with parameter $\kappa = 2$. In the same work, Lawler-Schramm-Werner showed that the peano curve associated with a uniformly random spanning tree converges to SLE_8.

A successful essay will review the proof of these results as well as discuss more recent developments on scaling limit results for loop-erased random walks.

Relevant Courses

Essential: Advanced Probability, Percolation

Useful: Schramm-Loewner Evolutions, Stochastic Calculus

References

[1] O. Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math*, 2000.

[2] G. F. Lawler, O. Schramm, W. Werner. Conformal invariance of planar loop-erased random walks and uniform spanning trees. *Ann. Probab.* 2004.

[3] D. Zhan. Loop-erasure of planar Brownian motion. *Comm. Math. Phys*, 2011.

[4] G. F. Lawler, F. Viklund. Convergence of loop-erased random walk in the natural parametrization, *arXiv preprint*, 2016.

46. Random Walk on Super Critical Percolation Clusters ....
Dr J. P. Miller

Perform bond percolation on $\mathbb{Z}^d$ (which means that each edge is kept independently with probability p). The critical probability is defined to be the supremum of values of p for which there almost surely does not exist an infinite connected component in the resulting random graph. We now fix $p > p_c$ and consider the unique infinite cluster $\mathcal{C}$ of bond percolation with parameter p. One way of probing the geometry of $\mathcal{C}$ is to perform a simple random walk on it, which is a process that at each time step jumps to a uniformly chosen neighbor (in the graph $\mathcal{C}$) of its current position. Barlow obtained Gaussian upper and lower bounds on the transition density for the continuous time walk and a few years later, Berger and Biskup and independently Matthieu and Piatnitski proved that for almost every percolation configuration the path of the walk suitably rescaled converges weakly to that of non-degenerate, isotropic Brownian motion. A successful essay should give an account of these developments and include proofs (or overviews of proofs) of the important results.

Relevant Courses

Essential: Advanced Probability, Percolation

References

[1] M. T. Barlow. Random walks on supercritical percolation clusters. *Ann. Probab.* 2004.

[2] N. Berger and M. Biskup. Quenched invariance principle for simple random walk on percolation clusters *Probab. Theory and Related Fields* 2007.

[3] P. Mathieu and A. Piatnitski. Quenched invariance principles for random walks on percolation clusters. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* 2007.

47. Random Walks on Height Functions .... Professor J. R. Norris

According to Donsker's invariance principle, any zero-mean, finite-variance random walk on the integers converges weakly under diffusive scaling to a Brownian motion. The diffusivity of the limit Brownian motion is simply the one-step variance of the random walk. The essay will examine the phenomenon of convergence to Brownian motion in a more general setting.

Suppose we are given a finite bipartite graph G with edge set E. Let us say that a function $f : G \to \mathbb{Z}$ is a height function if $|f(x) - f(y)| = 1$ whenever $(x, y) \in E$, and say that two height functions f and g are neighbours if $|f(x) - g(x)| = 1$ for all $x \in G$. Consider the random walk $(F_n)_{n\geq 0}$ on the set of height functions, that is, the random process which moves in each time step from its present state to a randomly chosen neighbour.

The aim of the essay is to show that the average height process $\bar{F}_n = \frac{1}{|G|}\sum_{x\in G} F_n(x)$ converges under diffusive scaling to a Brownian motion and to determine, at least in some special cases, the diffusivity of the limit. See Chapter 7 in [2] for an introduction to diffusion approximation. Ideas from [3] on correctors may also be useful. Some aspects of this essay may be open problems. Original work will receive special credit but is not necessary for an essay of Distinction standard.

Relevant Courses

Essential: None

Useful: Advanced Probability

References

[1] E. Boissard, S. Cohen, Th. Espinasse & J. Norris. Diffusivity of a random walk on random walks. Random Structures Algorithms 47 (2015), no. 2, 267–283.

[2] S.N. Ethier & T.G. Kurtz. *Markov processes: characterization and convergence.* Wiley (2005).

[3] M.J. Luczak & J.R. Norris. Averaging over fast variables in the fluid limit for Markov chains: application to the supermarket model with memory. Ann. Appl. Probab. 23 (2013), no. 3, 957–986.

48. Brownian Motion on a Riemannian Manifold .... Professor J. R. Norris

Brownian motion on a Riemannian manifold is the unique random process which satifies the two conditions that it is a martingale and that its quadratic variation is given by the metric tensor. Properties of this process are then closely related to both local and global properties of the manifold.

The essay will present an account of one or more constructions of Brownian motion on a Riemannian manifold and will discuss ways to characterize Brownian motion in terms of discrete approximations, as a Markov process, using stochastic differential equations, and via the heat equation. Then some further topics can be chosen in which the behaviour of Brownian motion is analysed. Examples of such topics are: recurrence and transience, behaviour under projections, Brownian bridge and geodesics, long-time behaviour, the case of Lie groups.

The nature of this essay is a synthesis of material in a well developed field. Given the availability of many relevant sources, special credit will be given for an attractive and coherent account.

Relevant Courses

Essential: None

Useful: Advanced Probability, Stochastic Calculus and Applications, Differential Geometry

References

[1] Elworthy, K. D. Stochastic differential equations on manifolds. London Mathematical Society Lecture Note Series, 70. Cambridge University Press, Cambridge-New York, 1982

[2] Emery, Michel. Stochastic calculus in manifolds. Universitext. Springer-Verlag, Berlin, 1989.

[3] Grigor'yan, Alexander. Heat kernel and analysis on manifolds. AMS/IP Studies in Advanced Mathematics, 47. American Mathematical Society, Providence, RI; International Press, Boston, MA, 2009

[4] Hsu, Elton P. Stochastic analysis on manifolds. Graduate Studies in Mathematics, 38. American Mathematical Society, Providence, RI, 2002.

49. Optimal Allocation in Sequential Multi-armed Clinical Trials with a Binary Response ....

Dr D. Robertson and Dr S. S. Villar

Before a novel treatment is made available to the wider public, clinical trials are undertaken to show that the treatment is safe and efficacious. Multi-armed trials, i.e. trials in which multiple treatment options are simultaneously considered, are increasingly being used to speed up the drug development process [1]. In such trials, sequential changes to the allocation probabilities for the different treatments can be made based on the accrued data, in order to achieve certain objectives. Efficiency and ethical goals create trade-offs that are well studied in sequential two-armed trials [2]. In the multi-armed setting, solutions that preserve statistical power while optimising an ethical measure are complex [3, 4, 5] and far less studied.

This essay could focus on cases in which optimal allocation admits a closed-from solution, explaining their derivation and discussing the limitations of the required assumptions. Alternatively, the focus could be on the numerical implementation of optimisation algorithms to solve cases in which a closed-formed expression is intractable, and analysing the properties of the resulting optimal allocation designs through computer simulations. The possibility of exploring different optimisation criteria and their solutions is encouraged.

Relevant Courses

Essential: None

Useful: Statistics in Medical Practice, Topics in Convex Optimisation

References

[1] G. Baron, F. Perrodeau, I. Boutron, P. Ravaud (2013). Reporting of analyses from randomized controlled trials with multiple arms: a systematic review. *BMC Medicine*, 11:84.

[2] W. F. Rosenberger, N. Stallard, A. Ivanova, C. N. Harper and M. L. Ricks (2001). Optimal Adaptive Designs for Binary Response Trials. *Biometrics*, 57:909–913.

[3] Y. Tymofyeyev, W. F. Rosenberger and F. Hu (2007). Implementing Optimal Allocation in Sequential Binary Response Experiments. *JASA*, 102(477):224–234.

[4] O. Sverdlov and W. F. Rosenberger (2013). On Recent Advances in Optimal Allocation Designs in Clinical Trials. *Journal of Statistical Theory and Practice*, 7:753–773.

[5] .S Villar., J. Bowden. and J. Wason. (2015) Multi-armed Bandit Models for the Optimal Design of Clinical Trials: Benefits and Challenges. *Statistical Science*, 30(2): 199–215

50. The EM and k-means Algorithms ....

Dr M. Gataric, Dr J. Jankova and Professor R. J. Samworth

Clustering, a canonical example of unsupervised learning, is one of the most common tasks of exploratory data analysis, with applications in a huge variety of scientific domains, including machine learning, image analysis, bioinformatics and many others (e.g. [1], Chapter 9). Two classical algorithms for this task are the Expectation–Maximisation (EM) and k-means algorithms ([2], [3]). Despite their enormous popularity, however, historical attempts to explain their behaviour have provided only limited insights ([4], [5], [6]), and, at least until very recently, both methods have been shrouded in mystery in terms of understanding when and why they work. Many of the difficulties stem from the fact that the underlying optimisation problems these algorithms attempt to solve are non-convex. However, in the last couple of years, there has been considerable progress towards understanding these algorithms ([7], [8], [9]).

After a brief historical review and description of the types of problem to which these algorithms can be applied, this essay would focus on these recent developments in understanding the EM and k-means algorithms. There is considerable scope for the candidate to explore their performance in new models, either theoretically or empirically.

Relevant Courses

Essential: None

Useful: Bayesian Modelling and Computation, Modern Statistical Methods

References

[1] Bishop, C. M. (2006) *Pattern Recognition and Machine Learning.* Springer Science+Business Media, New York.

[2] Dempster, A. P., Laird, N. M. and Rubin, D. B. (1977) Maximum likelihood from incomplete data via the EM algorithm. *J. Roy. Statist. Soc., Ser B*, **39**, 1–38.

[3] Lloyd, S. (1982) Least squares quantization in PCM. *IEEE Trans. Inf. Th.*, **28**, 129–137.

[4] Wu. C. F. J. (1983) On the convergence properties of the EM algorithm. *Ann. Statist.*, **11**, 95–103.

[5] Milligan, G. W. (1980) An examination of the effect of six types of error perturbation on fifteen clustering algorithms. *Psychometrika*, **45**, 325–342.

[6] Pollard, D. (1981) Strong consistency of K-means clustering. *Ann. Statist.*, **9**, 135–140.

[7] Balakrishnan, S., Wainwright, M. J. and Yu. B. (2017) Statistical guarantees for the EM algorithm: from population to sample-based analysis. *Ann. Statist.*, **45**, 77–120.

[8] Dwivedi, R., Ho, N., Khamaru, K., Jordan, M. I., Wainwright, M. J. and Yu, B. (2018) Singularity, misspecification, and the convergence rate of EM. `https://arxiv.org/abs/1810.00828`.

[9] Lu, Y. and Zhou, H. H. (2016) Statistical and computational guarantees of Lloyd's algorithm and its variants. `https://arxiv.org/abs/1612.02099`.

51. Applications of Random Matrix Theory in Statistics .... Professor R. J. Samworth and Dr Z. Zhu

Large random matrices occur in many areas of modern Statistics. Examples include estimators of high-dimensional covariance matrices and their inverses (e.g. [1], [2]), sparse Principal Components Analysis ([3], [4]) and random projections ([5], [6], [7]), among many others. Concentration inequalities and spectral properties often lie at the heart of their analysis, and results on rates of convergence etc. may be heavily dependent on the choice of matrix norm.

This essay will need to cover some of the relevant background in random matrix theory (e.g. [8], [9]), but should focus on how these results are applied in statistical contexts.

Relevant Courses

Essential: None

Useful: Modern Statistical Methods, Topics in Statistical Theory

References

[1] Bickel, P.J. and Levina, E. (2008) Covariance regularization by thresholding. *Ann. Statist.*, **36**, 2577–2604.

[2] Cai, T. T., Liu, W. and Zhou, H. H. (2016) Estimating sparse precision matrix: optimal rates of convergence and adaptive estimation. *Ann. Statist.*, **44**, 455–488.

[3] Zou, H., Hastie, T. and Tibshirani (2006) Sparse principal components analysis. *J. Comp. Graph. Statist.*, **15**, 262–286.

[4] Wang, T., Berthet, Q. and Samworth, R. J. (2016) Statistical and computational trade-offs in estimation of sparse principal components. *Ann. Statist.*, **44**, 1896–1930.

[5] Candès, E. and Tao, T. (2006) Near optimal signal recovery from random projections: universal encoding strategies?. *IEEE Trans. Inform. Theory*, **52**, 5406–5425.

[6] Cannings, T. I. and Samworth, R. J. (2017) Random-projection ensemble classification. *J. Roy. Statist. Soc., Ser. B (with discussion)*, **79**, 959–1035.

[7] Gataric, M., Wang, T. and Samworth, R. J. (2018) Sparse principal component analysis via random projections. Paper available at
`https://arxiv.org/abs/1712.05630`.

[8] Davidson K. R. and Szarek, S. J. (2001) Local operator theory, random matrices and Banach spaces, Handbook of the geometry of Banach spaces, Vol. I, North-Holland, Amsterdam, 2001, pp. 317–366.

[9] Vershynin, R. (2012) Introduction to the non-asymptotic analysis of random matrices. *Compressed sensing*, pp.210–268, Cambridge University Press.

52. Estimation of Heterogeneous Treatment Effects .... Dr T. B. Berrett and Professor R. J. Samworth

In the potential outcomes model for causal inference ([1], [2]), we consider independent copies of quadruples (X, A, Y^0, Y^1), where X is a covariate taking values in $\mathbb{R}^d$, A is a treatment indicator taking values in $\{0, 1\}$, and Y^a is the observed response when $A = a$ (we do not observe Y^{1-a}). The main quantity of interest is the so-called *heterogeneous treatment effect* ([3], [4]), given by $\tau(x) := \mathbb{E}(Y^1|X = x) - \mathbb{E}(Y^0|X = x)$.

The main difficulty here is that we observe only one of the two potential outcomes Y^0 and Y^1, so estimation of $\tau(\cdot)$ is impossible without further assumptions. A standard approach is to assume that there are no unmeasured confounders, i.e. (Y^0, Y^1) and A are conditionally independent given X. This facilitates estimation, e.g. based on *propensity weighting* ([5]).

One may also consider estimation of integral functionals of $\tau(\cdot)$, for instance using higher-order influence functions ([6]). These ideas rely on restrictive assumptions on the support of the covariate being compact, however, and appropriate functions being bounded away from their extremes. Such restrictions could possibly be alleviated using recent developments in the theory of functional estimation ([7], [8]), and this may provide an interesting new direction for an ambitious candidate.

Relevant Courses

Essential: None

Useful: Modern Statistical Methods, Topics in Statistical Theory

References

[1] Neyman, J. (1923) Sur les applications de la théorie des probabilités aux experiences agricoles: Essai des principes. *Roczniki Nauk Rolniczych*, **10**, 1–51.

[2] Rubin, D. B. (1974) Estimating causal effects of treatments in randomized and nonrandomized studies. *J. Educational Psychology*, **66**, 688–701.

[3] Robins, J., Li, L., Tchetgen, E., van der Vaart, A. (2008) Higher order influence functions and minimax estimation of nonlinear functionals. *IMS Collections Probability and Statistics: Essays in Honor of David A. Freedman*, **2**, 335–421.

[4] Wager, S. and Athey, S. (2018) Estimation and inference of heterogeneous treatment effects using random forests. *J. Amer. Statist. Assoc.*, **113**, 1228–1242.

[5] Hirano, K., Imbens, G. W. and Ridder, G. (2003) Efficient estimation of average treatment effects using the estimated propensity score. *Econometrica*, **71**, 1161–1189.

[6] Robins, J. M., Li, L., Mukherjee, R., Tchetgen Tchetgen, E. and van der Vaart, A. (2017) Minimax estimation of a functional on a structured high-dimensional model. *Ann. Statist.*, **45**, 1951–1987.

[7] Berrett, T. B., Samworth, R. J. and Yuan, M. (2018) Efficient multivariate entropy estimation via k-nearest neighbour distances. *Ann. Statist.*, to appear.

[8] Berrett, T. B. and Samworth, R. J. (2018) Efficient functional estimation and the super-oracle phenomenon. *In preparation.*

53. Recent Developments in False Discovery Rate Control .... Dr R. D. Shah

Since its introduction in 1995, Benjamini and Hochberg's paper [1] introducing the False Discovery Rate (FDR) has now become perhaps the most cited statistics paper of the last 25 years. Whilst the method they propose for performing FDR control continues to be hugely popular, in recent years there have been a number of important developments that improve upon this in different settings, and multiple testing is currently a highly active area within methodological statistics.

One body of work has considered various forms of structure among the hypotheses [2–4]. For example, it could be the case that one hypothesis being false logically implies that another may be false. Another interesting line of work has looked at procedures that work with certain test statistics rather than simply thresholding p-values at an appropriate point ([5–7]).

This essay could review some of the innovations in multiple testing that have been introduced in recent years and compare them theoretically and / or empirically. Another option would be to combine or slightly modify existing procedures or ideas with the aim of obtaining improved performance in certain settings.

Relevant Courses

Essential: Modern Statistical Methods

References

[1] Benjamini, Y. and Hochberg, Y. (1995) Controlling the false discovery rate: a practical and powerful approach to multiple testing. *Journal of the Royal Statistical Society, Series B*, **57**, 289–300.

[2] Li, A. & Barber, R. F. (2017) Multiple testing with the structure adaptive Benjamini–Hochberg algorithm. *arXiv:1606.07926*

[3] Ramdas, A. et al. (2017) A unified treatment of multiple testing with prior knowledge using the p-filter. *arXiv:1703.06222*

[4] Lei, L. et al. (2018) STAR: A general interactive framework for FDR control under structural constraints *arXiv:1710.02776*

[5] Barber, R. F. and Candes, E. J. (2015) Controlling the false discovery rate via knockoffs. *The Annals of Statistics* **43**, 2055–2085.

[6] Candes, E. et al. (2017) Panning for Gold: Model-X Knockoffs for High-dimensional Controlled Variable Selection *arXiv:1610.02351.*

[7] Cai, T. T., Sun, W., and Wang, W. (2017) CARS: Covariate Assisted Ranking and Screening for Large-Scale Two-Sample Inference.
Available at `http://www-bcf.usc.edu/ wenguans/Papers/CARS.pdf`.

54. Statistical Inference Using Machine Learning Methods ....

Dr R. D. Shah

The field of machine learning has much overlap with statistics, but has been predominantly concerned with prediction problems. Here it has had great successes with random forests, boosted trees, kernel machines and neural networks, among others, enjoying spectacular predictive performance in a variety of applications.

Statisticians, on the other hand, are often also concerned with parameter estimation for particular models, and uncertainty quantification. Recently, there has been a string of work aiming to harness the predictive power of machine learning methods for these more statistical goals. These can often lead to hypothesis tests with greater power than would be achievable using more classical tools, for example, or much more robust procedures whose validity holds across a far greater range of data-generating processes. The debiased Lasso [1] may be viewed as one example of this, where the Lasso is used to build confidence intervals for regression coefficients in high-dimensional settings. Other recent work considers statistical estimators based on random forests [2], deep neural networks [3] and regression splines [4]. The papers [5–7] propose more general procedures where prediction methods are plugged in, and study their properties.

Much of the work is tightly connected to the rich field of semiparametric statistics, and one option (among several) for the essay would be to review some of the papers referenced below and set some of the work within this context. Another option would be to focus more closely on a smaller subset of the papers and study some examples of the general methodology presented or propose some extensions.

Relevant Courses

Essential: Modern Statistical Methods

Useful: Topics in Statistical Theory, Statistical Learning in Practice

References

[1] Zhang, C.-H. & Zhang, S. (2014) Confidence intervals for low dimensional parameters in high dimensional linear models *J. Roy. Statist. Soc., Ser. B.*, 76(1), 217–242

[2] Athey, S. et al. (2018) Generalized Random Forests. *arXiv:1610.01271*

[3] Farrell, M. H. et al. (2018) Deep Neural Networks for Estimation and Inference: Application to Causal Effects and Other Semiparametric Estimands. *arXiv:1809.09953*

[4] Newey, W. & Robins, J. (2018) Cross-Fitting and Fast Remainder Rates for Semiparametric Estimation. *arXiv:1801.09138*

[5] Chernozhukov, V. et al. (2017) Double Machine Learning for Treatment and Causal Parameters. *arXiv:1608.00060.*

[6] Chernozhukov, V. et al. (2018) Generic Machine Learning Inference on Heterogeneous Treatment Effects in Randomized Experiments. *arXiv:1712.04802*

[7] Shah, R. D. & Peters, J. (2018) The Hardness of Conditional Independence Testing and the Generalised Covariance Meausre. *arXiv:1804.07203*

55. Model-Free No-Arbitrage Bounds ....

Dr M. Tehranchi

An important problem in financial risk management is to some how incorporate the information contained in quoted asset prices to find prices and hedging strategies for contingent claims. A classical approach is to introduce a model which is consistent with the observed prices, and then compute the prices and hedging strategies as predicted by the model. Unfortunately, more than one model may be consistent with a given set of data, and hence over-reliance on the predictions of a model exposes the practitioner to model-risk.

An alternative approach is compute upper lower bounds for contingent claim prices over *all* no-arbitrage models consistent with the data. Furthermore, it is often possible to compute the hedging strategies corresponding to the worst-case models.

When the observed market data consists of the price of a stock and a family of call option prices and the contingent claim is a barrier-style option on the stock, then the second problem can be reformulated in terms of a Skorokhod embedding problem. This technique has proved very effective. More recently, the developing theory of optimal martingale transport has provided a unified framework for studying the general model-free no-arbitrage pricing and hedging problem.

This essay should survey the recent literature on model-free pricing and hedging. Possible topics include the theory of Skorokhod embedding and the construction of certain optimal stopping times, the theory of optimal martingale transport, or a comparison between these two perspectives on the pricing and hedging problem

Relevant Courses

Essential: Advanced Financial Models, Stochastic Calculus & Applications

Useful: Advanced Probability

References

[1] M. Beiglböck, P. Henry-Labordére, F. Penkner. Model-independent bounds for option prices-a mass transport approach. *Finance and Stochastic* 17: 477–501. (2013)

[2] A. Galichon, P. Henry-Labordére, N. Touzi. A stochastic control approach to no-arbitrage bounds given marginals, with an application to lookback options. *Annals of Applied Probability* 24: 312–336. (2014)

[3] D. Hobson. The Skorokhod Embedding Problem and Model-Independent Bounds for Option Prices. *Paris-Princeton Lectures on Mathematical Finance 2010*, Lecture Notes in Mathematics 2003: 267–318. (2011)

[4] J. Obłój. The Skorokhod embedding problem and its offspring. *Probability Surveys* 1:321–392. (2004)

56. Polynomial Preserving Processes ....
Dr M. Tehranchi

A real-valued Markov process X is polynomial preserving if the function u defined by

$$u(t, x) = \mathbb{E}[f(X_t)|X_0 = x]$$

is a polynomial in x for all t whenever f is a polynomial. There is growing interest in modelling financial quantities with such processes since the computations involved in pricing certain derivative contracts are reasonably tractable.

This essay will survey the literature on polynomial preserving processes and related variants. Focus can be on the mathematical properties, such as characterisations of their generators, or can be on a particular application in finance, exploring their advantages and disadvantages compared to other modelling frameworks.

Relevant Courses

Essential: Advanced Financial Models, Stochastic Calculus & Applications

Useful: Advanced Probability

References

[1] S. Cheng and M. Tehranchi. Polynomial term structure models.
`http://arxiv.org/abs/1504.03238` (2016)

[2] Ch. Cuchiero, M. Keller-Ressel, and J. Teichmann. Polynomial processes and their applications to mathematical finance. *Finance and Stochastics* **16**: 711-740 (2012)

[3] D. Filipović and M. Larsson. Polynomial preserving diffusions and applications in finance. *Finance and Stochastic* **20**: 931972 (2016)

57. Precision Higgs Mass Predictions in Minimal Supersymmetry ....
Professor B. C. Allanach

The Minimal Supersymmetric Standard Model (MSSM) is still regarded by many to be an attractive TeV-scale extension to the Standard Model (however, in this essay, it is not necessary to review supersymmetry or the MSSM at all). The prediction of a Higgs boson whose properties match those of the experimentally discovered particle is an obvious priority. The calculation of its mass in particular has rather large radiative corrections, and is calculated to a relatively high order in perturbation theory, in various different schemes and approximations. It is a subject of active research as to which scheme or approximation links the Higgs boson mass prediction most precisely to the rest of the model.

The purpose of this essay is to find out and present the issues in the precision Higgs mass prediction in the MSSM, while providing an overall context.

The first half of the essay will set the scene in terms of experimental data for the Higgs boson discovery, and some analytical predictions for the lightest CP even Higgs boson mass prediction in terms of the other parameters of the model. The second half should address the important issues coming from approximations and different schemes, along with current attempts to address them, and their short-comings.

Relevant Courses

Essential: Quantum Field Theory, Standard Model, Particles and Symmetries, Advanced Quantum Field Theory

References

[1] D. de Florian *et al.* [LHC Higgs Cross Section Working Group], arXiv:1610.07922 [hep-ph].

[2] P. Kant, R. V. Harlander, L. Mihaila and M. Steinhauser, JHEP **1008** (2010) 104 arXiv:1005.5709 [hep-ph].

[3] R. V. Harlander, J. Klappert and A. Voigt, arXiv:1708.05720 [hep-ph].

[4] S. P. Martin, Adv. Ser. Direct. High Energy Phys. **21** (2010) 1 [Adv. Ser. Direct. High Energy Phys. **18** (1998) 1] [hep-ph/9709356].

(and references therein)

58. Edge-Turbulence Interaction and the Generation of Sound .... Dr L. J. Ayton

An unavoidable source of noise occurs when hydrodynamic pressure fluctuations move over a surface and encounter sudden changes to that surface, resulting in the fluctuations refracting and scattering into acoustic waves. A simple example occurs when a turbulent boundary layer above an aerodynamic wing encounters the sharp trailing edge.

This essay will discuss early analytical models of turbulence-edge interaction noise [1-4] then present ideas for how to update these models to include more realistic effects, for example the inclusion of surface roughness which could occur due to the inevitable damage of surfaces over time [5].

Relevant Courses

Essential: None

Useful: Perturbation Methods

References

[1] M. S. Howe (1978) A review of the theory of trailing edge noise. Journal of Sound and Vibration.

[2] B. Noble (1958) Methods Based on the Wiener-Hopf Technique for the Solution of Partial Differential Equations.

[3] D. G. Crighton & F. G. Leppington (1970) On the scattering of aerodynamic noise. Journal of Fluid Mechanics.

[4] D. S. Jones (1972) Aerodynamic sound due to a source near a half-plane. IMA Journal of Applied Mathematics.

[5] S. Glegg & W. Devenport (2009) The far-field sound from rough-wall boundary layers. Proceedings of the Royal Society.

59. Extrema in Gaussian Random Fields as a Proxy for Galaxy Clustering . Dr T. Baldauf

In the past two decades our picture of the Universe has been dramatically refined by observational campaigns that led to the measurement of the temperature fluctuations in the Cosmic Microwave Background (CMB) and the discovery of the accelerated expansion using supernovae. These new insights led to new ideas about the origin and evolution of the Universe and raised a plethora of new questions. For instance, what are the processes that seeded the rich structures that we observe today and what is causing the Universe to accelerate?

Large Scale Structure (LSS), the distribution of matter and galaxies in the late time Universe has been shown to have the potential to answer some of these questions. For instance, LSS is able to put an upper bound on the total neutrino mass, it can rule out some modifications of gravity and constrain the properties of dark energy models. Extracting these signals from upcoming galaxy surveys is a non-trivial task: the observable galaxies, which form in virialized clumps of dark matter (haloes), are an imperfect tracer of the matter distribution.

In the simplest model the galaxy two correlation function $\xi_{\rm gg}$ is assumed to be a linearly biased version of the (unobservable) matter correlation function $\xi_{\rm gg} = b_1^2 \xi_{\rm mm}$. This model ignores the finite size of the collapsed dark matter haloes and the specific locations of their formation sites. A simple, yet pheneomenologically interesting extension to this model is to assume that dark matter haloes form from maxima in the initial Gaussian density field. This so called peak model [2] makes distinct predictions for the shape of the baryon acoustic oscillation feature in the correlation function, the motion of haloes and their stochasticity.

In this essay you will first understand how the rich structure in the Universe arises from small, Gaussian fluctuations [1]. You will then focus on the properties of maxima in the Gaussian initial conditions (Lagrangian space) and describe their clustering properties [2,3,4] and stochasticity [5]. Finally you will explore how these properties translate into the evolved late time galaxy distribution [6,7].

Relevant Courses

Essential: Cosmology

Useful: Advanced Cosmology, Quantum Field Theory

References

[1] V. Desjacques, D. Jeong and F. Schmidt, “Large-Scale Galaxy Bias,” Phys. Rept. **733** (2018) 1 [arXiv:1611.09787].

[2] J. M. Bardeen, J. R. Bond, N. Kaiser, and A. S. Szalay, “The Statistics of Peaks of Gaussian Random Fields”, Astrophys. J. 304 (1986) 1561

[3] V. Desjacques, “Baryon acoustic signature in the clustering of density maxima,” Phys. Rev. D **78** (2008) 103503 doi:10.1103/PhysRevD.78.103503 [arXiv:0806.0007].

[4] T. Baldauf, S. Codis, V. Desjacques and C. Pichon, “Peak exclusion, stochasticity and convergence of perturbative bias expansions in 1+1 gravity,” Mon. Not. Roy. Astron. Soc. **456** (2016) no.4, 3985 [arXiv:1510.09204].

[5] T. Baldauf, U. Seljak, R. E. Smith, N. Hamaus and V. Desjacques, “Halo stochasticity from exclusion and nonlinear clustering,” Phys. Rev. D **88** (2013) no.8, 083507 [arXiv:1305.2917].

[6] V. Desjacques, M. Crocce, R. Scoccimarro and R. K. Sheth, “Modeling scale-dependent bias on the baryonic acoustic scale with the statistics of peaks of Gaussian random fields,” Phys. Rev. D **82** (2010) 103529 [arXiv:1009.3449].

[7] T. Baldauf and V. Desjacques, “Phenomenology of baryon acoustic oscillation evolution from Lagrangian to Eulerian space,” Phys. Rev. D **95** (2017) no.4, 043535 [arXiv:1612.04521].

60. Dualities and the Equivalence of Physical Theories ....

Dr J. N. Butterfield

A duality in physics is like a ‘giant symmetry’. In short: while a symmetry maps a state of the system into an appropriately related state (namely, one with the same values for a salient set of physical quantities): in a duality, an entire theory is mapped into another appropriately related theory. The important cases are those in which:
(i): a strong coupling regime of the first theory is mapped onto a weak coupling regime of the second theory; so that a problem that is hard in the first can be addressed, maybe solved, by solving an easier problem in the second; and-or
(ii) the theories seem to be about very disparate systems and-or phenomena, which suggests there are some deep links underneath the apparent differences: (the currently most famous example of this being gauge/gravity duality).
There are several good recent philosophical discussions of both aspects (i) and (ii) (which of course overlap). For a sample that addresses (i), cf. the papers in [1]. For a sample that addresses (ii), cf. the papers in [2].

This essay is about (ii): specifically, how to make precise—and so assess as true or false—the idea that two dual theories are ‘the same theory’. So the essay calls for some general account of what a duality is. The philosophical literature has suggestions about this, with an eye on topic (ii). The articles tend to treat several examples, and to compare the differences between dual theories with different fixings of a gauge. A sample of papers is [3]. The essay also calls for some account of the individuation of physical theories, i.e. an account of ‘theory’ precise enough to underwrite judgments of sameness and difference. This is clearly a topic for which one looks to logic and philosophy to supply an account. Indeed, these disciplines have considered various proposals about this, wholly independent of considerations about dualities in physics: the topic is called ‘theoretical equivalence’. This tradition has recently been revived by appeal to category theory. For a sample of recent papers, cf. [4].

The aim of the essay is to assess the various proposals (a) about what a duality is, and (b) about the equivalence of theories, so as to get a considered verdict on whether two given dual theories are 'the same theory'. Of course, the essay need not defend, or adopt, a single proposal (a). Similarly, it need not defend, or adopt, a single proposal (b). It can survey the different verdicts. (Nor is it required that the verdict about equivalence should be the same for any two dual theories.)

Relevant Courses

Essential: None

Useful: Philosophical aspects of symmetry and duality.

References

[1]: Castellani, E. (2017). 'Duality and particle democracy'. *Studies in History and Philosophy of Modern Physics*, pp. 100-108: doi:10.1016/j.shpsb.2016.03.002
Dieks, D., Dongen, J. van, Haro, S. de (2015), 'Emergence in Holographic Scenarios for Gravity'. *Studies in History and Philosophy of Modern Physics*, **52**, pp. 203-216. doi: 10.1016/j.shpsb.2015.07.007; arXiv:1501.04278 [hep-th].
De Haro, S. and Butterfield, J.N. (2018). 'A Schema for Duality, Illustrated by Bosonization'. *Foundations of Mathematics and Physics One Century after Hilbert.* J. Kouneiher (Ed.), Springer: arxiv: 1707.06681; http://philsci-archive.pitt.edu/13229.
Read, J. (2016). 'The Interpretation of String-Theoretic Dualities'. *Foundations of Physics*, **46**, pp. 209-235.

[2]: De Haro, S. (2018). 'The Heuristic Function of Duality'. *Synthese*; https://doi.org/10.1007/s11229-018-1708-9. arXiv:1801.09095 [physics.hist-ph].
Huggett, N. (2017). 'Target space $\neq$ space'. *Studies in History and Philosophy of Modern Physics*, **59**, 81-88. doi:10.1016/j.shpsb.2015.08.007.
Read, J. and Møller-Nielsen, T. (2018). 'Motivating Dualities'. Forthcoming in *Synthese.* http://philsci-archive.pitt.edu/14663
Rickles, D. (2011). A philosopher looks at string dualities. *Studies in History and Philosophy of Modern Physics*, **42**, pp. 54-67.

[3]: Butterfield, J. (2018). 'On Dualities and Equivalences Between Physical Theories'. Forthcoming in *Space and Time after Quantum Gravity*, Huggett, N. and Wüthrich, C. (Eds.). arxiv:1806.01505;
De Haro, S. (2018a). 'Theoretical Equivalence and Duality'. available from J. Butterfield.
Rickles, D. (2017). 'Dual theories: 'same but different' or different but same'?' *Studies in History and Philosophy of Modern Physics*, **59**, 62-67. doi: 10.1016/j.shpsb.2015.09.005.
Teh, N. and Tsementzis, D. (2017). Theoretical equivalence in classical mechanics and its relationship to duality. *Studies in History and Philosophy of Modern Physics*, **59**, pp. 44-54.

[4]: Halvorson, H. and Tsementzis, D. (2015). Categories of scientific theories. Forthcoming in *Categories for the Working Philosopher*, edited by E. Landry.
http://philsci-archive.pitt.edu/11923/
Glymour, C. (2013). 'Theoretical Equivalence and the Semantic View of Theories'. *Philosophy of Science*, **80**, pp. 286-297.

Weatherall, J. (2015). Categories and foundations of classical fields. Forthcoming in *Categories for the Working Philosopher*, edited by E. Landry. http://philsci-archive.pitt.edu/11587/; arxiv: 1505.07084.
Weatherall, J. (2016). Are Newtonian gravitation and geometrized Newtonian gravitation theoretically equivalent? *Erkenntnis*, **81**, pp. 1073-1091: http://philsci-archive.pitt.edu/11575/

61. Symmetry and Symplectic Reduction ....
Dr J. N. Butterfield

Symplectic reduction is a large subject in both classical and quantum mechanics. The course, 'Philosophical Aspects of Symmetry and Duality', will give an introduction to the classical aspects. This introduction will start from Noether's theorem in a classical Hamiltonian framework, and then explain (in terms of modern differential geometry) the ideas of: Lie group actions; the co-adjoint representation of a Lie group G on the dual $\mathfrak{g}^*$ of its Lie algebra $\mathfrak{g}$; Poisson manifolds (a mild generalization of symplectic manifolds that arise when one quotients under a symmetry); conserved quantities as momentum maps. With these ideas one can state the main theorems about symplectic reduction. The course will focus on the *Lie-Poisson* reduction theorem, which concerns the case where the natural configuration space for a system is itself a Lie group G. This occurs both for the pivoted rigid body and for ideal fluids. For example, take the rigid body to be pivoted, so as to set aside translational motion. This will mean that the group G of symmetries defining the quotienting procedure will be the rotation group SO(3). But it will also mean that the body's configuration space is given by $G = \mathrm{SO}(3)$, since any configuration can be labelled by the rotation that obtains it from some reference-configuration. So in this example of symplectic reduction, the symmetry group acts on itself as the configuration space. Then the theorem says: the quotient of the natural phase space (the cotangent bundle on G) is a Poisson manifold isomorphic to the dual $\mathfrak{g}^*$ of G's Lie algebra. That is: $T^*G/G \cong \mathfrak{g}^*$. There are several 'cousin' theorems, such as the *Marsden-Weinstein-Meyer* theorem. Main texts for this material include [1]. The course's exposition will use [2].

The essay should, starting from this basis, expound one or other of the following two topics. (Taking on both would be too much.)
(A): The first topic is technical and concerns the application of these classical ideas to quantum theory: more specifically, the interplay between reduction and canonical quantization. Physically, this is a large and important subject, since it applies directly to some of our fundamental theories, such as electromagnetism and Yang-Mills theories. The essay can confine itself to the more general aspects: which are very well introduced and discussed by Landsman and Belot; [3].
(B): The second topic is more philosophical. It concerns the general question under what circumstances should we take a state and its symmetry-transform to represent the same state of affairs—so that quotienting under the action of the symmetry group gives a non-redundant representation of physical possibilities? This question can be (and has been) discussed in a wholly classical setting. Indeed, the prototype example is undoubtedly the question debated between Newton (through his *ammanuensis* Clarke) and Leibniz: namely—in modern parlance—whether one should take a solution of, say, Newtonian gravitation for N point-particles and its transform under a Galilean transformation to represent the same state of affairs. This topic is introduced by the papers in [4]. In particular, Dewar discusses how, even when we are sure that a state and its symmetry-transform represent the same state of affairs, quotienting can have various disadvantages.

Relevant Courses

Essential: None

Useful: Symmetries, Fields and Particles; Philosophical Aspects of Symmetry and Duality.

References

[1]: R. Abraham and J. Marsden (1978), *Foundations of Mechanics*, second edition: Addison-Wesley; V. Arnold (1989), *Mathematical Methods of Classical Mechanics*, Springer, (second edition); J. Marsden and T. Ratiu (1999), *Introduction to Mechanics and Symmetry*, second edition: Springer-Verlag.

[2]: J. Butterfield (2006) On Symmetries and Conserved Quantities in Classical Mechanics, in W. Demopoulos and I. Pitowsky (eds.), *Physical Theory and its Interpretation*, Springer; 43 - 99; Available at: http://arxiv.org/abs/physics/0507192; J. Butterfield (2006). On Symplectic Reduction in Classical Mechanics, in J. Earman and J. Butterfield (eds.) *The Handbook of Philosophy of Physics*, North Holland; 1 - 131. Available at: http://arxiv.org/abs/physics/0507194.

[3]: G. Belot (1998), 'Understanding electromagnetism', *British Journal for the Philosophy of Science* **49**, p. 531-555; G. Belot (2003), 'Symmetry and gauge freedom', *Studies in History and Philosophy of Modern Physics* **34** 189-225;
N. Landsman (2006), 'Between Classical and Quantum', Section 4. in J. Earman and J. Butterfield (eds.) *The Handbook of Philosophy of Physics*, North Holland; 1 - 131. Available at: http://arxiv.org/abs/physics/0507194.
N. Landsman (2017). *Foundations of Quantum Theory*: Sections 5.6-5.12. Springer. Open access: downloadable at: https://link.springer.com/book/10.1007/978-3-319-51777-3

[4]: Four papers by G. Belot: (2000), 'Geometry and motion', *British Journal for the Philosophy of Science* **51**, p. 561-596; (2001), 'The principle of sufficient reason', *Journal of Philosophy* **98**, p. 55-74; (2003), 'Notes on symmetries', in Brading and Castellani (ed.s) (2003), pp. 393-412. (2013), 'Symmetry and equivalence', in R. Batterman (ed.), *Oxford Handbook of Philosophy of Physics* Oxford University Press, 2013. All Belot papers are available at:
https://sites.google.com/site/gordonbelot/home/papers-etc
Caulton, A. (2015). 'The Role of Symmetry in the Interpretation of Physical Theories'. *Studies in History and Philosophy of Modern Physics*, **52**, pp. 153-162. N. Dewar (2017) 'Sophistication about symmetries', *British Journal for the Philosophy of Science*: available at: https://academic.oup.com/bjps/advance-article-abstract/doi/10.1093/bjps/axx021/4111183

62. Viscoelastic Instabilities in Soft Matter .... Professor M. E. Cates

In many non-Newtonian fluids, such as entangled polymer solutions or liquid crystals, nonlinear instabilities can arise in shear flow at zero Reynolds number (that is, in the absence of the inertial nonlinearity of the Navier Stokes equation) [1]. This is often because these materials have a nonlinear relation between viscoelastic stress and strain rate history, which replaces the linear relation between viscous stress and strain rate in a Newtonian fluid [2]. Instabilities include shear-banding – in which a homogeneously sheared fluid separates into zones of high and low viscosity [3]– and tumbling, in which a fluid with orientational order (such as a liquid crystal) undergoes periodic or chaotic dynamics of the axis of order [4]. The essay should give a synoptic survey of such viscoelastic instabilities before focusing in more depth on one or two of them, freely chosen.

Relevant Courses

Essential: None

Useful: Theoretical Physics of Soft Condensed Matter; Slow Viscous Flow; Hydrodynamic Stability

References

[1] R. G. Larson, Instabilities in viscoelastic flows, Rheological Acta 31, 213-263 (1992)

https://doi.org/10.1007/BF00366504

[2] S. M. Fielding and M. E. Cates, Rheology of giant micelles, Advances in Physics 55, 799-879 (2006)

https://doi.org/10.1080/00018730601082029

https://arxiv.org/abs/cond-mat/0702047

[3] S. M. Fielding, Complex dynamics of shear banded flows, Soft Matter 3, 1262-1279 (2007)

https://doi.org/10.1039/b707980j

[4] G. Rienaker, M. Kroger, S. Hess, Chaotic orientational behavior of a nematic liquid crystal subjected to a steady shear flow Phys. Rev. E 66, 040702 (2001)

https://doi.org/10.1103/PhysRevE.66.040702

63. Mixing Efficiency in Stratified Fluids .... Professor C. P. Caulfield

Stratified fluids (i.e. fluids with density differences) are ubiquitous in the environment and industry. A particularly important issue is the extent to which turbulent motions irreversibly change the density distribution within the flow (i.e. 'mix' the fluid). Since mixing in a stratified fluid inevitably changes the potential energy of the flow, it is of great interest to understand the efficiency of the mixing, i.e. the proportion of the work done on the flow that leads to irreversible mixing. This essay should investigate the underlying issues of the energetics of high Reynolds number stratified flows, and then consider some of the various approaches to describe and parameterize stratified mixing processes, many of which have quite surprising and counter-intuitive aspects.

Relevant Courses

Essential: None

Useful: Fluid Dynamics of the Environment, Hydrodynamic Stability

References

[1] H. J. S. Fernando (1991): 'Turbulent mixing in stratified fluids.' *Ann. Rev. Fluid Mech.* **23** 455-493.

[2] P. F. Linden (1979): 'Mixing in stratified fluids.' *Geophys. Astrophys. Fluid Dyn.* **13** 3-23.

[3] G. N. Ivey, K. B. Winters & J. R. Koseff (2008): 'Density stratification, turbulence, but how much mixing?' *Ann. Rev. Fluid Mech.* **40** 169-184.

[4] T. R. Osborn (1980): ‘Estimates of the local-rate of vertical diffusion from dissipation measurements.’ *J. Phys. Oceanogr.* **10** 83-89.

[5] K. B. Winters, P . N. Lombard, J. J. Riley, & E. A. D’Asaro (1995): ‘Available potential-energy and mixing in density-stratified fluids.’ *J. Fluid Mech.* **289** 115-128.

64. Instability and Perturbation Growth in Stratified Shear Flows .... Professor C. P. Caulfield

Stratified shear flows, where both the fluid density and the velocity vary with height, are extremely common both in the environment and in industrial contexts. It is of great practical importance to understand how such flows undergo the transition to turbulence, as turbulence typically hugely increases mixing, transport and dissipation within such flows. It is commonly believed that ‘normal’ mode flow instabilities play a central role in this transition process, and the conventional argument is that the ‘most unstable’ normal mode will dominate the nonlinear evolution of the flow, and hence lead the flow to transition. However, the underlying linearized operator is non-normal, and so it is possible for substantial transient growth of perturbations to occur. Although this has been widely studied in unstratified flows, [1,2] the transient behaviour of stratified flows has been much less-studied. Also, stratified shear flows are prone to multiple, qualitatively different primary and secondary instabilities (particularly when the density distribution develops sharp interfaces [3]) and it appears that the transition to turbulence is typically associated with **secondary** instabilities which only develop once the primary instability has saturated [4]. There are also several interesting mathematical issues about the ‘optimal’ measures of perturbation growth to use, as the potential energy as well as the kinetic energy of the perturbation varies in a stratified flow [5], and this essay could approach the general issue of perturbation growth in stratified shear flows from a variety of mathematical and computational directions.

Relevant Courses

Essential: Hydrodynamic Stability

Useful: Fluid Dynamics of the Environment

References

[1] P. J. Schmid 2007. Non-modal stability theory. *Ann. Rev. Fluid Mech.* **39** 129-162.

[2] R. R. Kerswell 2018. Nonlinear nonmodal stability theory. *Ann. Rev. Fluid Mech.* **50** 319-345.

[3] H. Salehipour, C. P. Caulfield & W. R. Peltier 2016. Turbulent mixing due to the Holmboe wave instatbility at high Reynolds number. *J. Fluid Mech.* **258** 255-285.

[4] W. R. Peltier & C. P. Caulfield 2003. Mixing efficiency in stratified shear flows. *Ann. Rev. Fluid Mech.* **35** 135-167.

[5] A. Kaminski, J. R. Taylor & C. P. Caulfield 2014. Transient growth in strongly stratified shear layers. *J. Fluid Mech.* **758** R4, 12 pages.

65. Recovering Quantum Information .... Dr N. Datta

A fundamental problem in quantum information theory is to determine how well lost information can be reconstructed. Crucially, the corresponding recovery operation should perform well without the knowledge of the information to be reconstructed. A useful figure of merit for the performance of such recovery operations is given by an entropic quantity called the *quantum conditional mutual information (QCMI).* An explicit characterisation of tripartite quantum states for which the QCMI is zero, and which hence saturate the powerful strong subadditivity inequality for the von Neumann entropy, has been obtained in [1]. This characterisation also yields a necessary and sufficient condition for quantum error correction. A tripartite quantum state for which the QCMI is zero, is said to form a so-called (short) quantum Markov chain. It is hence natural to expect that tripartite quantum states whose QCMI is small but non-zero are close to a quantum Markov chian. However, counterexamples of this intution were provided in [2], and the problem of characterising such states remained open for years. A crucial breakthrough on this problem was finally made by Fawzi and Renner in [3]. This was followed by further interesting developments (see e.g. [4], [5] and [6]) of the notions of recoverability of quantum information and approximate quantum Markov chains. This essay will entail a review of this very timely and interesting research area. Students are not expected to cover all the papers listed below.

Relevant Courses

Essential: Quantum Information Theory (Lent)

Useful: Markov Chains (Part IB)

References

[1] P. Hayden, R. Jozsa, D. Petz, and A. Winter, Structure of states which satisfy strong subadditivity of quantum entropy with equality, *Communications in Mathematical Physics, 246(2):359374, 2004.*

[2] B. Ibinson, N. Linden, and A. Winter, Robustness of quantum Markov chains, *Communications in Mathematical Physics, 277(2):289304, 2008.*

[3] O. Fawzi and R. Renner, Quantum conditional mutual information and approximate Markov chains, *Communications in Mathematical Physics, 340(2):575611, 2015.*

[4] D. Sutter, O. Fawzi and R. Renner, Universal recovery map for approximate Markov chains, *Proceedings of the Royal Society A, vol. 472, no. 2186, 20150623 2016.*

[5] F.G.S.L. Brandão, A. W. Harrow, J. Oppenheim, and S. Strelchuk, Quantum conditional mutual information, reconstructed states, and state redistribution, *Physical Review Letters, 115(5):050501, July 2015.*

[6] N. Datta and M.M. Wilde. Quantum Markov chains, sufficiency of quantum channels, and Rényi information measures, *Journal of Physics A vol. 48, no. 50, page 505301, November 2015.*

66. Polynomial Optimization on the Sphere .... Dr. H. Fawzi

The problem of maximizing a degree-four polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$ on the unit sphere $S^{n-1} = \{x \in \mathbb{R} : \|x\|_2 = 1\}$ is hard in general. A sequence of tractable *convex relaxations* can be used however to approximate the solution of such an optimization problem. More precisely there is a sequence $v_1 \geq v_2 \geq \cdots \geq p^* = \max_{x \in S^{n-1}} p(x)$ such that each v_k can be computed as a *semidefinite program* of size n^k, and where $v_k \to \max p$. The quantity v_k is defined as the smallest γ such that $\gamma - p$ is a sum-of-squares of polynomials of degree k modulo the sphere. The question of how fast the sequence (v_k) converges to $\max p$ is still not very well understood.

The objective of this essay will be to summarize the known results related to this question, and to explore the connections between this problem and the problem of separability testing in quantum information theory.

Relevant Courses

Essential: Topics in Convex Optimisation

References

[1] E. de Klerk, The complexity of optimizing over a simplex, hypercube or sphere: a short survey, http://www.optimization-online.org/DB_HTML/2006/09/1475.html

[2] A. Harrow, A. Natarajan, X. Wu, An Improved Semidefinite Programming Hierarchy for Testing Entanglement, Communications in Mathematical Physics, https://arxiv.org/abs/1506.08834

67. 'Moisture Modes' and the Tropical Atmosphere .... Professor P. H. Haynes

The tropical atmosphere is driven by radiative heating towards a state which is relatively warm at the surface and relatively cold at altitude and as a result strong convection develops, manifested by tall cumulus clouds. However the entire tropics is not in a state of active convection, but instead there is strong spatial variation at scales ranging from those of individual clouds to scales of hundreds or thousands of kilometres with large regions of active convection adjacent to large regions where convection is rare or even absent altogether.

One particular feature of the tropical atmosphere is the so-called 'Madden-Julian oscillation' (MJO). This is not a regular oscillation, but a quasi-random variation in convection and in dynamical quantities such as wind and temperature, in which a region of active convection appears over the tropical Indian Ocean, drifts eastward into the western Pacific and then diminishes in strength over the eastern Pacific. The time between successive appearances of the active convection is typically 30-60 days. Most of the global circulation models used for climate prediction give very poor simulations of the MJO, suggesting that they poorly represent the physical processes that are responsible for it, probably because it depends on quite subtle interactions between convecting and non-convecting regions and between large scales and the scales of the weather systems within which active convection is embedded.

Given the complication of the tropical atmosphere – the range of spatial and temporal scales and the importance of cloud-scale processes including interactions between clouds and radiation – it might seem that simple mathematical models would have limited relevance. However, provided the need for crude but simple representations of cloud-scale processes is accepted, relatively simply models can capture some of the important interactions between these processes and the large-scale dynamics and provide genuine insight into ways in which the representation of tropical circulations in global climate models might be improved.

One particular class of models that has been studied over the last 15 years or so, and is now being argued to provide a basis for understanding the MJO, are models in which include the simple fluid dynamical equations (e.g. as represented by the 'shallow-water equations') together with a moisture field that is transported with the fluid and affects the fluid dynamics by determining the heating. An interesting limit is when the fluid dynamics is treated as quasi-steady and the entire time evolution is controlled by the moisture field. In this limit simple wave motion is sometimes possible and these waves are described as 'moisture modes'. There is now quite a large literature on 'moisture modes' and their behaviour according to different dynamical formulations (e.g. incorporating different physical processes) and there are also several papers which discuss possible moisture-mode models for the MJO.

An essay on this topic should start by surveying some of the basic papers that have studied moisture modes in different forms, trying to present a unified summary of the important features of the behaviour and how it depends on the physical ingredients incorporated in the model. The essay might then move on to discuss in more detail the extent to which moisture modes provide an explanation for the MJO and how these simple models might be used to advance understanding of the MJO and to improve its representation in climate models. (But a student writing this essay might choose to focus on other topics, such as the way in which convection-scale processes are represented in the simple models, or the extent to which moisture modes, or simple models that allow moisture modes along other sorts of behaviour, are useful to understand other aspects of the tropical atmosphere.)

Relevant papers are listed below. The introduction to [1] provides a short overview of work on moisture modes and cites several relevant papers. [2] is an early paper that considers a relevant simple model and identifies moisture-mode behaviour. [3] is a relatively mathematical paper that may be easier for a Part III Mathematics student to read than a paper written for atmospheric scientists. [4] and [5] are papers that propose moisture-mode models for the MJO. [6] is a paper that argues on the other hand that the physical processes incorporated into the models described in [4] and [5] may not be relevant to the MJO in the real atmosphere and offers an alternative.

Relevant Courses

Essential: An undergraduate course in fluid dynamics

Useful: Hydrodynamic Stability, Fluid Dynamics of Climate (Neither is absolutely essential, but a any student who is considering choosing this essay and who is NOT intending to take Fluid Dynamics of Climate is advised – and welcome – to discuss with the setter.)

References

[1] Sugiyama, M., 2009: The moisture mode in the quasi-equilibrium tropical circulation model: Part I: Analysis based on the weak temperature gradient approximation. J. Atmos. Sci., 66, 1507-1523.

[2] Sobel, A. H., J. Nilsson, and L. M. Polvani, 2001: The weak temperature gradient approximation and balanced tropical moisture waves. J. Atmos. Sci., 58, 3650-3665.

[3] Sukhatme, J., 2013: Low-frequency modes in an equatorial shallow-water model with moisture gradients. Quart. J. Roy. Meteor. Soc., 140, 1838-1846.

[4] Sobel, A., and E. Maloney, 2013. Moisture modes and the eastward propagation of the MJO. J. Atmos. Sci., 70, 187-192.

[5] Adames, A., and D. Kim, 2016: The MJO as a dispersive, convectively coupled moisture wave: Theory and observations. J. Atmos. Sci., 73, 913-941.

[6] Fuchs, Z. and D. J. Raymond, 2017: A simple model of intraseasonal oscillations. J. Adv. Model. Earth Syst., 9, 1195-121.

68. Viscoplastic Convection ....

Dr D. R. Hewitt

Viscoplastic fluids are a branch of non-Newtonian fluids characterised by a 'yield stress', below which they behave as a rigid solid, and above which they flow like a viscous fluid. A great array of materials, from mud and industrial waste to toothpaste and whipped cream, exhibit some degree of viscoplasticity, and they can behave in quite a different manner to their Newtonian counterparts. The interaction of buoyancy forces with viscoplasticity leads to a fascinating array of dynamics and flow patterns and plays an important role in a wide range of problems, from magmatic flows and mud volcanism to the cooking of porridge on a hot plate.

This essay will begin with a review of the classical models for viscoplastic fluids, including shear-thinning 'regularisations' and the simple Bingham model. The essay should then discuss the linear-stability problem, with a thorough review and discussion of how the introduction of a yield stress stabilizes the problem to linear perturbations, but not to non-linear ones. After reviewing this core material, the candidate should explore extensions, including at least some of: the dynamics of different canonical convective systems (e.g. Rayleigh–Bénard, Rayligh–Taylor, heated side walls); analytically tractable convective solutions; numerical advances and challenges in modelling more complex flows; experimental techniques and observations; and physical applications and implications.

Relevant Courses

Essential: None

Useful: Slow Viscous Flow, Fluid Mechanics of the Solid Earth, Perturbation Methods, Hydrodynamic Stability

References

[1] Zhang, J., Vola, D. and Frigaard, I.A. Yield stress effects on Rayleigh–Bénard convection, *J. Fluid Mech.* **556**, 389-419 (2006)

[2] Balmforth, N,J. and Rust, A.C. Weakly nonlinear viscoplastic convection, *J. Non-Newtonian Fluid Mech.* **158**, 36-45 (2009)

[3] Karimfazli, I. and Frigaard, I.A. Natural convection flows of a Bingham fluid in a long vertical channel, *J. Non-Newtonian Fluid Mech.*, **201**, 39-55 (2013)

[4] Patel, N. and Ingham, D.B. Analytic solutions for the mixed convection flow of non-Newtonian fluids in parallel plate ducts *Int. Comm. Heat Mass Transfer*, **21**, 79-84 (1994)

[5] Balmforth, N.J., Frigaard, I.J. and Ovarlez, G. Yielding to stress: recent developments in viscoplastic fluid mechanics, *Ann. Rev. Fluid Mech.* **46**, 121-146 (2014)

69. Hydrodynamics of Mid-Ocean Ridges ....
Dr D. R. Hewitt

Roughly 30% of the heat lost from the oceanic crust is removed in mid-ocean-ridge hydrothermal systems, which consist of convectively circulating oceanic fluid below the sea bed and above a magmatic heat source. Hydrothermal circulation at mid-ocean ridges affects the composition of the crust and the ocean, and contains a wealth of complex fluid mechanics.

This essay will explore the complex hydrodynamics of mid-ocean-ridge (MOR) systems. Given the complexity and number of different processes involved, the essay is fairly open-ended, but should certainly include as core material a review of the main processes involved in MOR hydrothermal systems and a discussion of the mathematics of convection in porous media, which provides the key mechanism of heat transfer. In light of the presence of both heat and salt in the system, the concept and basic mathematics of double-diffusive convection should also be discussed. Beyond this, there are a number of directions that candidates could develop this essay, for example by exploring: further details and more advanced analysis of convection and double-diffusion in porous media; models of the mechanics of heat transport and crystallisation in the magma chamber; details of compaction in the upper mantle and magmatic system ('viscous compaction' models); or the induced flow in 'black smoker' plumes above the sea floor.

Relevant Courses

Essential: Fluid Mechanics of the Solid Earth

Useful: Slow Viscous Flow, Fluid Mechanics of the Environment, Hydrodynamic Stability

References

[1] Lovell, R.P., Crowell, B.W., Lewis, K.C. and Liu, L. Modeling multiphase, multicomponent processes at oceanic spreading centers, *Chapter in: Geophysical Monograph Series 178* (2008)

[2] Alt, J. C. Subseafloor processes in mid-ocean ridge hydrothermal systems, *Chapter in: Geophysical Monograph Series 91* (1995)

[3] Hewitt, D.R., Neufeld, J.A. and Lister, J.R. Ultimate Regime of High Rayleigh Number Convection in a Porous Medium, *Phys. Rev. Lett.* **108**, 224503 (2012)

[4] Fontaine, F.J. and Wilcock, W.S.D. Two-dimensional numerical models of open-top hydrothermal convection at high Rayleigh and Nusselt numbers, *Geochem., Geophys., Geosys.* **8** (2007)

[5] Huppert. H.E. Geological fluid mechanics. *Chapter in: Perspectives in Fluid Dynamics: a Collective Introduction to Current Research,* 393-446. CUP (2000)

[6] Spiegleman, M. Flow in deformable porous media. Part 1 Simple analysis. *J. Fluid Mech.* **247**, 17-38 (1993)

70. Deterministic Descriptions for Large-scale Behaviour of Stochastic Models ....
Dr R. Jack

Small particles (size $< 1\mu$m) in water move by Brownian motion. They follow apparently random paths, that can be described by stochastic differential equations or Langevin equations.

However, if one considers many such particles, their density ρ follows a deterministic diffusion equation $(\partial\rho/\partial t) = D\nabla^2\rho$. How can it be that particles move randomly but the density follows a deterministic equation?

At a field-theoretic level, this behaviour can be understood within a saddle-point approximation. For a more precise statement, one should realise that there is a finite probability that the particle density does *not* follow the diffusion equation, but this probability tends to zero as the number of particles $N \to \infty$. The nature of this limit is described by the probabilistic theory of *large deviations*, which quantifies the probability of very rare events, as reviewed in [1]. The application of this theory to systems of many particles is the content of *macroscopic fluctuation theory* [2].

This essay will discuss the extent to which macroscopic fluctuation theory resolves our original puzzle – how can large-scale behaviour can be described deterministically, when the microscopic equations of motion are stochastic? Two related directions are the use of stochastic partial differential equations (with small noise) to describe the behaviour of the density as a function of time [3], and the rigorous mathematical theory of hydrodynamic limits [4].

Relevant Courses

Essential: Theoretical Physics of Soft Condensed Matter.

Useful: Stochastic Calculus and Applications.

References

[1] H. Touchette, *The large deviation approach to statistical mechanics*, Phys. Rep. **478**, 1 (2009)

[2] L. Bertini *et al.*, *Macroscopic fluctuation theory*, Rev. Mod. Phys. **87**, 593 (2015).

[3] D. S. Dean, *Langevin equation for the density of a system of interacting Langevin processes*, J. Phys. A **29**, L613 (1996)

[4] M. H. Duong, A. Lamacz, M. A. Peletier, and U. Sharma, *Variational approach to coarse-graining of generalized gradient flows*, Calc. Var. Partial Diff. Equ. **56**, 100 (2017)

71. Dynamical Phase Transitions into Absorbing States ....

Dr R. Jack

The theory of equilibrium phase transitions is based on probabilities of *configurations* of a system, such as an Ising model. In *dynamical phase transitions*, one concentrates instead on the *dynamical trajectories* (or paths) that a system follows, as a function of time. This opens up a range of new possibilities.

Some famous experiments in 2005 revealed a surprising new phase transition [1] in a system of small particles (size $\sim$ 0.2mm). In one of the phases ("active phase") the particles move around, apparently at random. In the other phase ("inactive"), the particles move along periodic trajectories; they never collide with each other, so they keep returning to the same places. These experiments were understood in terms of a theoretical model [2] in which a similar transition takes place. It is now understood that this transition is accompanied by a diverging correlation length (and an associated correlation time), similar to equilibrium phase transitions. However, the theory of equilibrium phase transitions does not apply here.

There are field theories that account for the dynamical trajectories of the system and can describe critical points of this type, but there are significant open questions about the universality class that is relevant for these models and experiments [3]. Recent studies have also uncovered new properties of the critical point, such as "hyperuniformity" of the inactive phase [4].

This essay will review the theory of these dynamical phase transitions, the phenomena that occur near the critical point, and the connection with experimental and computer simulation data. The theoretical treatments are based on field theories or stochastic partial differential equations – understanding which terms are "relevant" in these theories can be much more tricky than the standard approaches that work for equilibrium phase transitions.

Relevant Courses

Essential: Theoretical Physics of Soft Condensed Matter

Useful: Statistical Field Theory

References

[1] D. J. Pine, J. P. Gollub, J. F. Brady and A. M. Leshansky, *Chaos and threshold for irreversibility in sheared suspensions*, Nature **438**, 997 (2005)

[2] L. Corte, P. M. Chaikin, J. P. Gollub and D. J. Pine, *Random organization in periodically driven systems*, Nature Physics **4**, 420 (2008)

[3] E. Tjung and L. Berthier, *Criticality and correlated dynamics at the irreversibility transition in periodically driven colloidal suspensions*, J. Stat. Mech. (2016), 033501,

[4] D. Hexner and D. Levine, *Noise, Diffusion, and Hyperuniformity*, Phys. Rev. Lett. **118**, 020601 (2017).

72. Renormalization Group Study of Constrained Hamiltonians .... Dr R. Jack and Dr E. Tjhung

At criticality, or second order phase transition point, the correlation length of a physical system diverges as power law $\xi \sim |T - T_c|^{-\nu}$ (T_c being the critical temperature). $\nu > 0$ is a critical exponent whose value is independent of the microscopic details of the system. In other words, at criticality, many different physical systems share the same critical exponents (universality class). It has been thought for a long time that different universality classes can be determined solely from their symmetries [1]. For instance, liquid/gas critical point and Ising model at critical temperature correspond to up/down symmetry and para-ferromagnetic transition corresponds to rotational symmetry. However it was recently discovered that symmetry alone may not uniquely define the universality class of the system. For instance, one can imagine a Hamiltonian with the same rotational symmetry as in a para-ferromagnetic transition but with an additional constraint $\nabla \times \mathbf{m} = 0$ [2]. This constraint gives a new fixed point in the renormalization group flow, which indicates a different universality class from the unconstrained Hamiltonian (even though they have the same symmetry). Topologically, the constraint $\nabla \times \mathbf{m} = 0$ suppresses any vortices/spiral defects which may be created in the system.

Similarly, one can also imagine the same Hamiltonian with another constraint $\nabla \cdot \mathbf{m} = 0$ [3] and this will give yet another different fixed point in the renormalization group flow. Topologically speaking, this constraint will kill off any hedgehog defects in the system. In this essay, we will investigate how these two different pictures will fit together.

Relevant Courses

Essential: Statistical Field Theory (Part III).

Useful: Theoretical Physics of Soft Condensed Matter (Part III), Statistical Physics (Part II).

References

[1] M. Kardar, *Statistical Physics of Fields*, Cambridge University Press (2007)

[2] X. Ma and E. Tjhung, Banana and pizza-slice-shaped mesogens give a new constrained ferromagnet universality class, arXiv:1807.07039.

[3] O. I. Motrunich and A. Vishwanath, Emergent Photons and New Transitions in the O(3) Sigma Model with Hedgehog Suppression, arXiv:cond-mat/0311222.

73. The Grasshopper Problem .... Professor A. P. A. Kent

Bell's theorem shows that locally causal hidden variable theories cannot reproduce all the predictions of quantum theory for measurements on separated entangled states, for example the singlet state of two spin $\frac{1}{2}$ particles. Bell inequalities give bounds on correlations obtainable from locally causal hidden variable theories for specific classes of measurements, which are violated by some quantum states.

Recently, Kent and Pitalua-Garcia considered measurements of pairs of separated spin $\frac{1}{2}$ particles about two spin axes $\underline{a}, \underline{b}$, chosen randomly subject to the constraint that $\underline{a}.\underline{b} = \cos(\theta)$. They showed that Bell inequalities can be proven for such measurements, for a wide range of θ. However, the strongest possible bounds are not known for most values of θ.

There is a natural geometric and pictorial interpretation of this question in terms of colourings of the Bloch sphere and "jumps" through an angle θ, in which one can picture the jump as if made by a grasshopper that landed randomly on the sphere and jumps through fixed angle θ in a random direction. This interpretation is intriguing enough in its own right to motivate considering the problem outside the original context of quantum theory and Bell inequalities. For example, Goulko and Kent considered the "grasshopper problem" on the plane, and were able to prove some analytic results and obtain extensive numerical evidence about the form of apparently (near-)optimal colourings.

An ideal essay would review the literature to date and produce interesting extensions of the known results, most likely through new numerical solutions.

Relevant Courses

Essential: None

Useful: Part II Principles of Quantum Mechanics, Part II Quantum Information and Computation

References

[1] Bloch-sphere colorings and Bell inequalities, Kent, Adrian and Pitalúa-García, Damián, Physical Review A, **90**, 062124, (2014).

[2] The grasshopper problem, Goulko, Olga and Kent, Adrian, Proc. R. Soc. A, **473**, 2207, 20170494, (2017).

74. Does the Miles-Howard Theorem Have any Relevance to Turbulent Stratified Shear Flows? ....

Professor R. R. Kerswell

The Miles-Howard theorem (e.g. Howard 1961) is a classical result in stably-stratified flows which gives sufficient conditions for linear stability of a steady, inviscid, unidirectional stratified shear flow. If the local Richardson number (Ri) of the flow (a non-dimensional parameter measuring the relative strength of the stable stratification to the shear) is everywhere larger than $\frac{1}{4}$, the Miles-Howard theorem asserts that the flow has to be linearly stable. While this result has been derived in very idealised circumstances (e.g. the absence of fluid viscosity), the prediction of stability for $Ri > \frac{1}{4}$ seems to hold some greater truth. Many observations seem to show turbulent flows operating at or just below $Ri = \frac{1}{4}$ suggesting that this value may actually approximate some sort of nonlinear stability barrier and that the turbulence is marginally stable. This part III essay would explore the literature (starting with the references below) to discuss the evidence for these statements.

Relevant Courses

Essential: Fluids II, Hydrodynamic Stability, Methods

Useful: Asymptotics, Dynamical Systems

References

[1] Howard, L.N. 1961 "Note on a paper by John W. Miles" *J. Fluid Mech.* **10**, 509-512.

[2] Hazel, P. 1972 "Numerical studies of inviscid stratified shear flows" *J. Fluid Mech.* **51**, 39-61.

[3] Thorpe, S.A. & Liu, Z. 2009 "Marginal stability?" *J. Phys. Ocean.* 2373-2381.

[4] Smyth, W.D. & Moum, J.N. 2013 "Marginal instability and deep cycle turbulence in the eastern equatorial Pacific Ocean" *Geophys. Res. Lett.* **40**, 6181-6185.

[5] Salehipour, H., Peltier, W.R. & Caulfield, C.P. 2018 "Self-organized criticality of turbulence in strongly stratified mixing layers" *arXiv:1809.03039.*

75. Self-Sustaining Processes in 2 Dimensional Shear Flows ....

Professor R. R. Kerswell

Understanding how and when simple shear flows break down to turbulence is an ongoing problem of huge importance in understanding the environment and industrial applications. Much of the recent progress made in understanding transition to turbulence in linearly-stable shear flows has followed from uncovering a 'self-sustaining process' which stops solutions unconnected to the basic shear flow from decaying away through viscous damping. This process, first found by Waleffe (1997) and later recognised as the vortex-wave-interaction theory of Hall and Smith, is inherently 3-dimensional. In 2 dimensions, transition is usually only found when there is a linear instability of the basic shear flow (e.g. channel flow). However, there are situations where non-decaying solutions unconnected to a linearly-stable shear flow exist indicating a different type of self-sustaining process. Deguchi & Walton (2013) describe just such a situation - axisymmetric annular sliding Couette flow - and identify these non-decaying states via tracking bifurcations from other parts of parameter space. They also describe the asymptotic structure of the non-decaying solutions. The aim of this part III essay would be to try to deconstruct this 2D self-sustaining process into its component parts with the aim of generating the equivalent cartoon to Waleffe's (1997) figure 1 for his 3D process (streamwise rolls generate streaks which become unstable to waves which then nonlinearly self-interact to drive the original rolls). This essay is a great opportunity for somebody interested in mathematical fluid dynamics and asymptotics to immerse themselves in a very current topic.

Relevant Courses

Essential: Asymptotics, Fluids II, Hydrodynamic Stability, Methods

Useful: Dynamical Systems

References

[1] Deguchi, K. & Walton, A.G. 2013 "Axisymmetric travelling waves in annular sliding Couette flow at finite and asymptotically large Reynolds number" *J. Fluid Mech.* **720**, 582-617.

[2] Haberman, R 1972 "Critical layers in parallel flows" *Stud. Appl. Maths* **51**, 139-161.

[3] Walton, A.G. 2003 "The nonlinear instability of thread-annular flow at high Reynolds number" *J. Fluid Mech.* **477**, 227-257.

[4] Waleffe, F. 1997 "On self-sustaining process in shear flows" *Phys. Fluids*, **9**, 883-900.

76. Thermocapillary Instabilities ....
Dr K. Kowal

An interface between two immiscible fluids is subject to interfacial, or surface, tension, which may depend on various scalar fields, such as the temperature and solute concentration, as well as on the concentration of surfactants, or compounds that decrease surface tension. When surface tension depends on the temperature, gradients along the interface induce shear stresses that result in thermocapillary fluid flow. Thermocapillary flows are ubiquitous in nature and industry, such as in crystal growth, welding, the manufacture of silicon wafers, electron beam melting of metals and the rupture of thin films. In many of these, the transport of heat can increase significantly as a result of additional mixing processes triggered by thermocapillary instabilities. The onset of these instabilities can be examined using linear stability theory.

The essay should review the mechanisms of the instability using linear stability theory as well as characterise the different modes of instability that occur in *static* and *dynamic* thermocapillary liquid layers. The candidate may choose to examine the long-wave evolution of thermocapillary waves, their nonlinear dynamics, or the effects of surfactants and thermophoresis (the Soret effect). The exact direction of the essay depends on the interests of the candidate.

Relevant Courses

Essential: Undergraduate Fluid Mechanics

Useful: Hydrodynamic Stability, Slow Viscous Flow

References

[1] Pearson, J. R. A. (1958) On convection cells induced by surface tension. *J. Fluid Mech.* **4** (5), 489–500.

[2] Smith, M. K. and Davis, S. H. (1983) Instabilities of dynamic thermocapillary liquid layers. Part 1. Convective instabilities. *J. Fluid Mech.* **132** 119–144.

[3] Davis, S. H. (1987) Thermocapillary instabilities. *Ann. Rev. Fluid Mech.* **19**, 403–435

[4] Oron, A., Davis, S. H. and Bankoff, S. G. (1997) Long-scale evolution of thin liquid films. *Rev. Mod. Phys.* **69** (3)

[5] Oron, A. (2000) Nonlinear dynamics of three-dimensional long-wave Marangoni instability in thin liquid films. *Phys. Fluids* **12**, 1633

[6] Saenz, P. J., Valluri, P., Sefiane, K., Karapetsas, G., and Matar, O. K. (2013) Linear and nonlinear stability of hydrothermal waves in planar liquid layers driven by thermocapillarity. *Phys. Fluids* **25**, 094101

77. Capillary Retraction of Viscous Fluid Sheets .... Professor J. R. Lister

When a hole forms in a soap film, a capillary force/length 2γ acts on the edge of the hole, pulling it outward and rapidly increasing the size of the hole. Early work by Taylor, Culick and then Keller looked at the inviscid dynamics appropriate to watery soap films. More recently, there has been considerable interest in viscous dynamics, whether for rupture of very viscous films, retraction of the edge of a sheet of molten glass, or expansion of the 'hole' in the sheet of external fluid between coalescing bubbles or drops.

The essay should review the development of theoretical modelling for the viscous case (i.e. large, though possibly finite, Ohnesorge number) and discuss the range of applications. It should include a derivation for the case of radial sheet flow analogous to that provided for axisymmetric extensional flow in lectures. Some angles to explore in the literature might include the differences between two-dimensional and radial flow, the effect of the initial sheet thickness profile, the boundary condition at the edge, and the role of inertia when the Ohnesorge number is large but finite. The essay should conclude with a discussion of the extent to which the theory explains experimental observations, and where the theory needs to be developed further.

Relevant courses

Essential: Slow Viscous Flow

References

Debrégeas, G., Martin, P. & Brochard-Wyart, F. 1995 Viscous bursting of suspended films. *Phys. Rev. Lett.* **75**, 3886–3889.

Brenner, M. P. & Gueyffier, D. 1999 On the bursting of viscous films. *Phys. Fluids* **11**, 737–739.

Savva, N. & Bush, J. W. M. Viscous sheet retraction. 2009 *J. Fluid Mech.* **626**, 211–240.

Munro, J.P., Anthony, C.R., Basaran, O.A. & Lister, J.R. 2015 Thin-sheet flow between coalescing bubbles. *J. Fluid Mech.* **773**, R3.

Munro, J.P. & Lister, J.R. 2018 Capillary retraction of the edge of a stretched viscous sheet. *J. Fluid Mech.* **844**, R1.

Paulsen, J. D., Carmigniani, R., Burton, A. Kannanand J. C. & Nagel, S. R. 2014 Coalescence of bubbles and drops in an outer fluid. *Nat. Commun.* **5**.

Bird, J. C., de Ruiter, R., Courbin, L. & Stone, H. A. 2010 Daughter bubble cascades produced by folding of ruptured thin films *Nature*, 465, 759.

78. The Shape of a (Viscous) Chocolate Fountain .... Professor J. R. Lister

In a chocolate fountain, molten chocolate flows down over a vertical stack of dome-shaped tiers of increasing size (Try googling images for 'chocolate fountain'). The flow on each tier can be described very simply using lubrication theory. The chocolate flows over the circular edge of each tier to form a thin axisymmetric curtain of falling fluid, which falls until it lands on the next tier below. Observation shows that the curtain contracts inwards as it falls, with radius looking almost linear as a function of height. The theory for an inviscid curtain of fluid is well-established from the study of 'water bells', but molten chocolate is viscous!

This essay would likely take the form of a mini-project to calculate the shape of a falling very viscous Newtonian fluid curtain by using the equations of viscous extensional flow, as adapted to the axisymmetric geometry. The equations of axisymmetric shell theory in elasticity would be a useful comparison. Inertia should be neglected, but surface tension is (probably) relevant. The problem differs from tube-drawing in that radial and axial variations are comparable. Further references and guidance are available on request.

Relevant courses

Essential: Slow Viscous Flow

References

Townsend, A.K. & Wilson, H.J. 2015 The fluid dynamics of the chocolate fountain *Euro. J. Phys.* **37**.

Ribe, N.M. 2002 A general theory for the dynamics of thin viscous sheets. *J. Fluid Mech.* **457**, 255.

Audoly, B. & Pomeau, Y. 2010 *Elasticity and geometry* OUP

79. Kinks with Long-Range Tails .... Professor N. S. Manton

Kinks, the basic type of topological soliton in one space dimension, are classical solutions of a scalar field theory with multiple vacua. Kinks usually have exponentially localised, short-range tails, but recently there has been an effort to better understand kinks having long-range tails with power-law fall-off. Such tails arise when the field is massless to the left or right of the kink. The interaction of a kink with another kink, or with an antikink, is tricky to calculate, because the long-range tail obeys a nonlinear equation. In this essay you should review how kinks with long-range tails arise, and the attempts to understand their interactions through numerical and analytical studies. A comparison with the easier case of kinks having short-range tails will be worthwhile.

Relevant Courses

Essential: None

Useful: Quantum Field Theory; Classical and Quantum Solitons

References

Kinks are described in most field theory books discussing solitons, e.g.

[1] N. Manton and P. Sutcliffe, Topological Solitons, Cambridge University Press, 2004,

[2] Y. M. Shnir, Topological and Non-topological Solitons in Scalar Field Theories, Cambridge University Press, 2018,

and the books by R. Rajaraman, T. Vachaspati, E. Weinberg etc. Kinks with long-range tails were considered by

[3] J. A. Gonzalez and J. Estrada-Sarlabous, Kinks in systems with degenerate critical points, Phys. Lett. **A140**, 189 (1989),

and for recent discussions, see

[4] N. S. Manton, Force between kinks with long-range tails, arXiv:1810.00788; Force between kink and antikink with long-range tails, arXiv:1810.03557,

[5] E. Belendryasova and V. A. Gani, Scattering of the ϕ^8 kinks with power-law asymptotics, Commun. Nonlinear Sci. Numer. Simulat. **67**, 414 (2019),

[6] I. C. Christov et al., Long-range interactions of kinks, arXiv:1810.03590.

80. Wave Attractors in Rotating and Stratified Fluids ....
Professor G. I. Ogilvie

Internal waves can propagate in rotating and/or stably stratified fluids (as often occur in astrophysical and geophysical settings) as a result of Coriolis and/or buoyancy forces. Their properties are radically different from those of acoustic or electromagnetic waves. The frequency of an internal wave depends on the direction of the wavevector but not on its magnitude. Waves of a given frequency follow characteristic paths through the fluid and reflect from its boundaries. In many cases the rays typically converge towards limit cycles known as wave attractors. One application of this finding is to tidally forced fluids in astrophysical and geophysical settings. If tidal disturbances are focused towards a wave attractor, this can lead to efficient tidal dissipation that in some cases is independent of the small-scale diffusive processes.

This essay should review the subject of internal wave attractors, including some of the more recent developments. Some simple explicit examples should be provided, which could involve original calculations. Topics that might be covered include:

1. The behaviour of rays for pure inertial waves in a uniformly rotating spherical shell.

2. The relation, if any, between the propagation of rays within a container and the existence of invisicid normal modes.

3. The consequences of a wave attractor for the decay rate of a free oscillation mode, or the dissipation rate of a forced disturbance, in the presence of a small viscosity.

4. The roles of nonlinearity and instability in wave attractors.

5. The relevance of wave attractors to tidal dissipation in astrophysical systems.

Relevant Courses

Essential: None

Useful: Astrophysical Fluid Dynamics

References

[1] Maas, L. R. M. & Lam, F.-P. A. (1995). *J. Fluid Mech.* **300**, 1

[2] Maas, L. R. M., Benielli, D., Sommeria, J. & Lam, F.-P. A. (1997). *Nature* **388**, 557

[3] Rieutord, M., Georgeot, B. & Valdettaro, L. (2001). *J. Fluid Mech.* **435**, 103

[4] Ogilvie, G. I. (2005). *J. Fluid Mech.* **543**, 19

[5] Jouve, L. & Ogilvie, G. I. (2014). *J. Fluid Mech.* **745**, 223

81. Eccentric Astrophysical Discs ....

Professor G. I. Ogilvie

Closed Keplerian orbits around a massive body are generally non-circular. A thin Keplerian disc may be composed of nested elliptical orbits whose eccentricity e and longitude of periapsis ϖ vary continuously with semi-major axis a and time t. The complex eccentricity is $e\ \exp(\mathrm{i}\varpi) = E(a,t)$.

When $|E|$ and $|\partial E/\partial \ln a|$ are sufficiently small, a linear evolutionary equation can be derived for the complex eccentricity, which determines how the shape of the disc propagates by means of pressure, viscosity, self-gravity and other collective effects that are weak compared to gravity. More generally, $E(a,t)$ satisfies a nonlinear evolutionary equation and the presence of eccentricity affects the transport of mass and angular momentum in the disc.

Eccentric discs are thought to exist in many astrophysical situations, including narrow planetary rings around Saturn and Uranus, protoplanetary discs around young stars, circumstellar discs around rapidly rotating Be stars, accretion discs around compact objects in close binary systems, circumbinary discs around binary black holes or young binary stars, and debris discs around white dwarfs.

This essay should discuss aspects of the dynamics and significance of eccentric discs in at least one of these areas of application. Apart from the derivation and interpretation of the evolutionary equation(s) for eccentric discs, theoretical topics that might be discussed include the stability of fluid flows with elliptical streamlines, the gravitational interaction of orbiting companions with a disc, and the numerical simulation of eccentric discs.

A selection of recent references is provided below; use of the NASA ADS archive to locate other relevant publications is recommended.

Relevant Courses

Essential: None

Useful: Astrophysical Fluid Dynamics, Dynamics of Astrophysical Discs, Planetary System Dynamics

References

[1] Barker, A. J. and Ogilvie, G. I. (2014). *Mon. Not. R. Astron. Soc.* **445**, 2637

[2] Miranda, R. and Rafikov, R. R. (2018). *Astrophys. J.* **857**, 135

[3] Ogilvie, G. I. and Barker, A. J. (2014). *Mon. Not. R. Astron. Soc.* **445**, 2621

[4] Teyssandier, J. and Ogilvie, G. I. (2016). *Mon. Not. R. Astron. Soc.* **458**, 3221

[5] Thun, D., Kley, W. and Picogna, G. (2017). *Astron. Astrophys.* **604**, 102

[6] Wienkers, A. F. and Ogilvie, G. I. (2018). *Mon. Not. R. Astron. Soc.* **477**, 4838

82. The BMS Group, the Soft Graviton and Gravitational Memory .... Professor M. J. Perry

The Bondi-Metzner-Sachs group is the group of asymptotic symmetries of a spacetime that is asymptotically flat. It should be thought of as a generalization of the Poincaré group of symmetries of Minkowski spacetime. A gravitational wave escaping to null infinity will generate a BMS transformation that can be interpreted as a form of gravitational memory. Curiously, the BMS group is also related to the notorious infrared divergences encountered in the quantum field theory of gravitation. Weinberg's soft graviton theorem explains how these divergences can be cancelled. The essay should explore and develop these concepts and show how they can regarded as different faces of the same fundamental idea.

Relevant Courses

Essential: General Relativity, Quantum Field Theory

Useful: Advanced Quantum Field Theory

References

[1] "Asymptotic symmetries in gravitational theory" R. Sachs, Phys.Rev. 128 (1962) 2851-2864.

[2] "Infrared photons and gravitons" Steven Weinberg, Phys.Rev. 140 (1965) B516-B524.

[3] "Lectures on the Infrared Structure of Gravity and Gauge Theory" Andrew Strominger, arXiv:1703.05448

83. Defining the Anderson-Bergmann Velocity Poisson Bracket .... J. B. Pitts

Do we yet know what a Poisson bracket ought to mean? The oft-cited 1951 Anderson-Bergmann paper on constrained Hamiltonian dynamics contains a little-noticed 'Poisson bracket' for velocities, $\{\dot{q}, F\} = \frac{d}{dt}\{q, F\}$. This bracket appears to yield only correct answers, to be indispensable (apart from a near-equivalent involving secret introduction of temporal smearing functions) for answering some important questions (such as the gauge in/covariance of Hamilton's equations and the canonical Lagrangian $p\dot{q}-H$), to violate the usual Poisson bracket product rule occasionally in order to satisfy the time differentiation product rule, and to have no clear mathematical basis.

Discuss these issues.

Relevant Courses

Essential: None

Useful: Hamiltonian General Relativity

References

Besides references from the useful course, see:

1. J. L. Anderson and P. G. Bergmann, *Physical Review* **83** (1951), p. 1018.
2. K. Sundermeyer, *Constrained Dynamics: With Applications to Yang–Mills Theory, General Relativity, Classical Spin, Dual String Model.* Springer, Berlin (1982).
3. L. Castellani, *Annals of Physics* **143** (1982), 357.
4. T. Thiemann, in D. J. W. Giulini, Claus Kiefer and C. Lämmerzahl, eds., *Quantum Gravity: From Theory to Experimental Search.* Springer, Berlin (2003), p. 41; gr-qc/0210094.
5. J. B. Pitts, *Classical and Quantum Gravity* **34** (2017), 055008, arXiv:1609.04812 [gr-qc].

84. Validation and Extensions of the Parabolic Equation Method in Random Media ....

Dr O. Rath Spivack

Problems of wave propagation in inhomogeneous media are difficult to solve analytically. Solutions based on the parabolic wave equation (PWE) method, have been used successfully in many cases, including the propagation of acoustic waves in the ocean, and acoustic or optical waves in turbulent atmosphere.

In the absence of exact solutions, a major challenge is the validation of different approximations, and evaluating their accuracy. Since the early work of Tappert [1], several developments have improved the efficiency of numerical implementation, for example through the introduction of split-step methods. Other developments have extended the scope of the approximation to include wide angle geometries and backscatter or reverberation, for example through higher order approximations and two-way parabolic equations. Further research has been devoted to dealing with the multiscale nature of the wave propagation problem when the wavelength is much smaller than the range over which signals are measured, and the scale of the inhomogeneities is comparable to the wavelength.

This essay should focus on propagation in random media. After an introduction to the PWE method and an overview of approximation methods, it should choose specific examples for which to explain ways of estimating accuracy and compare with other available approximation.

A few possible references are given below, but more specific references depending on the choice of focus will be provided.

Relevant Courses

Essential: Knowledge of the wave equation and basic concepts in wave propagation, from any course.

Useful: The Part III course Direct and Inverse Scattering of Waves

References

[1] F. D. Tappert, The Parabolic Approximation Method, in J. B. Keller, J. S. Papadakis, eds., Wave Propagation and Underwater Acoustics, Lecture Notes in Physics, vol. 70, Springer, New York, 1977

[2] J. F Lingevitch, M.D. Collins and M. J. Mills, A two-way parabolic equation that accounts for multiple scattering, JASA 112, 476 (2002)

[3] Kai Huang, Knut Solna, and Hongkai Zhao, Coupled Parabolic Equations for Wave Propagation, Methods Appl. Anal. Vol. 11, Number 3 (2004), 399-412.

[4] Guillaume Bal and Lenya Ryzhik, Time splitting for wave equations in random media, ESAIM: M2AN Volume 38, Number 6 (2004), 961 - 987

85. Krylov Subspace Methods for the Regularisation of Inverse Problems . . Dr O. Rath Spivack

Many inverse problems in mathematical physics can be formally expressed as

$$Ax = y \ , \tag{1}$$

where A is an operator from a normed vector space X into a normed vector space Y, $y \in Y$ is given data, often measured data, and $x \in X$ is the unknown. Usually such problems are ill-posed, and various methods are used to ensure existence and uniqueness, as well as stability of the solution, which is referred to as 'regularisation'.

Regularisation can in some cases be achieved by projection onto finite-dimensional subspaces $A_n \subset X$. Krylov subspace methods are iterative methods in which the solution is sought by successive approximations $x_n \in K_n$, where K_n is the Krylov subspace $span\{d, Bd, \cdots B^{n-1}d\}$, with B and d dependent on A and y. The Conjugate Gradient method and its variants are examples of Krylov subspace methods, and other have also been used, sometimes together with Tikhonov regularisation.

This essay should explore the regularising properties of Krylov subspace methods, focusing on some particular issues according to personal interests and background. This could also be, for example, applications of Krylov methods to practical inverse problems.

A few example references are given below, and more will be provided depending on the choice of focus.

Relevant Courses

Essential: Basic knowledge of linear analysis, from any course.

Useful: The Part III courses Inverse Problems in Imaging, Direct and Inverse Scattering of Waves, Mathematics of Image Reconstruction (non-examinable)

References

[1] B. Kaltenbacher. Regularization by projection with a posteriori discretization level choice for linear and nonlinear ill-posed problems. Inverse Problems, 16(5): 1523-1539, 2000.

[2] M. Hanke, The Minimal Error Conjugate Gradient method is a regularization method, Proc. of the American Math Soc (1995) **123**, 3487

[3] M. Hanke On Lanczos Based Methods for the Regularization of Discrete Ill-Posed Problems Hanke, M. BIT Numerical Mathematics (2001) 41, pp 1008-1018

[4] S. Gazzola, P. Novati and M. R. Russo, On Krylov projection methods and Tikhonov regularization, Electronic Transactions on Numerical Analysis (2015) 44, p. 83-123

86. Dimensional Reduction ....

Dr R. A. Reid-Edwards

String and Superstring theories exist in higher dimensional spacetimes. This requirement is forced upon us by the quantum consistency of these theories. One way to make contact with the four-dimensional physics of experience is to assume that these extra dimensions take the form of small compact geometries that are thus far beyond our ability to detect directly. The study of how lower dimensional effective theories depend on the details of the compact space and how one might realise known theories as compactifications of higher dimensional theories is the subject of dimensional reduction (often referred to as Kaluza-Klein theory). One of the most intriguing aspects is the realisation of non-gravitational physics, such as Yang-Mills theory, as purely gravitational physics in higher dimensions.

The utility of dimensional reduction extends far beyond the requirements of string theory. Dimensional reduction gives a way of understanding the existence of certain supergravity theories in terms of the dimensional reduction of a higher dimensional theory and gives a concrete way of constucting new supergravity theories from known ones in higher dimensions. More recently, the desire to understand certain supergravity theories as arising as compactifications of string theory have suggested the existence of string theory backgrounds which cannot be understood in terms of a wordsheet embedding into a conventional Riemannian manifold, pointing towards an intrinsically string-theoretic generalisation of Riemannian geometry.

The first part of the essay will include a study of the dimensional reduction of simple gravitational theories on tori. The essay may then proceed in many different directions. One possible direction is to study compactifications on manifolds with non-abelian isometries and the relationship with gauge theory and gauged supergravities in the lower dimensions. Simple twisted or Scherk-Schwarz compactifications might also be studied. The question of consistency, in the sense that solutions to the lower dimensional equations of motion lift to solutions of the higher dimensional theory could be dicussed. The interested essayist might choose to investigate the relationship between compactification on Tori and the duality symmetries of string theory.

Relevant Courses

Essential: Part III General Relativity, Part III Quantum Field Theory

Useful: Part III String Theory, Part III Advanced Quantum Field Theory

References

[1] C Pope, "Lectures on Kaluza Klein," (may be found at http://people.physics.tamu.edu/pope/)

[2] M. J. Duff, B. E. W. Nilsson and C. N. Pope, "Kaluza-Klein Supergravity," Phys. Rept. **130** (1986) 1.

[3] T. Ortin "Gravity and Strings," CUP.

[4] C. M. Hull and R. A. Reid-Edwards, "Flux compactifications of string theory on twisted tori," [hep-th/0503114].

[5] M. Cvetic, G. W. Gibbons, H. Lu and C. N. Pope, "Consistent group and coset reductions of the bosonic string," [hep-th/0306043].

87. Strong Cosmic Censorship in Asymptotically de Sitter Spacetimes ..... Dr J. E. Santos

All observed gravitational dynamics in our Universe appears consistent with Einstein's theory of General Relativity, possibly endowed with a positive cosmological constant. However, the appearance of Cauchy horizons in certain solutions of the Einstein equation signals a potential breakdown of determinism within GR - the future history of any observer that crosses such an horizon cannot be determined using the Einstein field equation and the initial data! Penroses Strong Cosmic Censorship conjecture proposes that solutions containing such horizons cannot be dynamically generated starting with generic initial data.

The essay should do a thorough review of the recent proposed scenarios for violating the Strong Cosmic Censorship conjecture, and should be written in a language accessible to other Part III students taking similar courses.

Relevant Courses

Essential: General Relativity and Black Holes

References

[1] V. Cardoso, J. L. Costa, K. Destounis, P. Hintz and A. Jansen, "Quasinormal modes and Strong Cosmic Censorship," Phys. Rev. Lett. **120**, no. 3, 031103 (2018)

[2] O. J. C. Dias, H. S. Reall and J. E. Santos, "Strong cosmic censorship: taking the rough with the smooth," JHEP **1810**, 001 (2018)

88. Confinement .... Dr D. B. Skinner

The microscopic degrees of freedom of QCD are quarks and gluons. In the world around us such particles are never seen individually, but are always trapped within composite hadrons such as protons, neutrons and pions. This phenomenon is not seen in perturbation theory, and a complete understanding remains one of the outstanding challenges of theoretical physics. This essay will explore aspects of confinement in various contexts, from solvable toy models such as the $\mathbb{CP}^n$ model in $d = 2$, Polyakov's confinement mechanism in the Abelian Higgs model in $d = 3$, to 't Hooft's picture of the $d = 4$ QCD vacuum as a dual superconductor.

Relevant Courses

Essential: Advanced Quantum Field Theory

Useful: Statistical Field Theory, The Standard Model.

References

[1] A. M. Polyakov, *Quark Confinement and Topology of Gauge Groups*, Nucl. Phys. B120, 429 (1977).

[2] G. 't Hooft, *Topology of the Gauge Condition and New Confinement Phases in Non-Abelian Gauge Theories*, Nucl. Phys. B190, 455 (1981).

[3] E. Witten, *Dynamical Aspects of QFT* in *Quantum Fields and Strings: A Course for Mathematicians*, vol. 2, AMS 1999.

[4] D. Tong, *Gauge Theory*, lecture notes available at
http://www.damtp.cam.ac.uk/user/tong/gaugetheory.html

89. Computational Complexity of Quantum Ball Permuting Model .... Dr S. Strelchuk

The study of the computational power of many physical systems including noninteracting fermions, noninteracting bosons, or anyons in a 2+1 dimensional quantum field theory [1-4] can be recast as a study of models of computation based on permuting distinguishable particles – much like quantum 'balls' in boxes [5].

Some of these systems turn out to be solvable or integrable, and thus regarded as simple from the mathematical physics point of view, but they are nevertheless interesting from the complexity-theoretic point of view: their computational power appears to lie between classical and quantum computing.

The essay should introduce the quantum ball permuting model [5] and explain how it models actual physical processes. It should then discuss the computational complexity of one or two physical systems [1-4] that can be described in this framework.

Relevant Courses

Recommended: Part III Quantum Computing

References

[1] Aaronson, Scott, and Alex Arkhipov. "The computational complexity of linear optics." Proceedings of the 43rd ACM symposium on Theory of computing. ACM, 2011.

[2] Bremner, Michael J., Richard Jozsa, and Dan J. Shepherd. "Classical simulation of commuting quantum computations implies collapse of the polynomial hierarchy." Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences. Vol. 467. No. 2126. The Royal Society, 2011.

[3] Havlicek, Vojtech, and Sergii Strelchuk. "Quantum Schur Sampling Circuits can be Strongly Simulated." Physical review letters 121.6 (2018): 060505

[4] Morimae, Tomoyuki, Keisuke Fujii, and Joseph F. Fitzsimons. "Hardness of classically simulating the one-clean-qubit model." Physical review letters 112.13 (2014): 130502.

[5] Aaronson, Scott, et al. "The computational complexity of ball permutations." Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing. ACM, 2017

90. Variational Hybrid Quantum-Classical Algorithms .... Dr S. Strelchuk

Modern quantum algorithms require computational resources which are currently beyond the reach of state of the art implementations. But even minimal quantum resources can be made useful if we use them in conjunction with powerful classical optimization methods. This approach has been exploited in Variational Quantum Eigensolver (VQE) [1]. It makes use of Ritzs variational principle to prepare approximations to the ground state and its energy. In this algorithm, the quantum computer is used to prepare a class of variational 'trial' states which are characterized by a set of parameters. Then, the expectation value of the energy is estimated and used by a classical optimizer to generate a new set of improved parameters which are then used to prepare the next iteration of trial states. The advantage of VQE over purely classical simulation techniques is that it is able to prepare trial states that cannot be generated by efficient classical algorithms.

This essay should discuss the algorithm and its applications [2-3].

Relevant Courses

Recommended: Part III Quantum Computing

References

[1] McClean, Jarrod R., et al. "The theory of variational hybrid quantum-classical algorithms." New Journal of Physics 18.2 (2016): 023023.

[2] Kandala, Abhinav, et al. "Hardware-efficient variational quantum eigensolver for small molecules and quantum magnets." Nature 549.7671 (2017): 242.

[3] O'Malley, P. J. J., et al. "Scalable quantum simulation of molecular energies." Physical Review X 6.3 (2016): 031007.

91. Lattice QCD and Hadron Spectroscopy .... Dr C. E. Thomas

Quantum chromodynamics (QCD) is a quantum field theory that exhibits many interesting phenomena such as asymptotic freedom and confinement. Moreover, it describes the strong interaction of particle physics, i.e. how quarks and gluons interact and give rise to the non-trivial structure and dynamics of hadrons. Recent observations of a number of 'exotic' structures have generated a lot of interest and hadrons are currently the subject of many theoretical and experimental investigations.

Computing the masses and other properties of hadrons within QCD is a long-standing challenge because the QCD coupling is strong in the relevant low-energy regime. Lattice QCD is a non-perturbative technique that enables first-principles computations of the properties of hadrons using Monte Carlo methods. This essay should give a brief introduction to lattice QCD and then discuss its application to calculating the spectra of hadrons.

Relevant Courses

Essential: Quantum Field Theory

Useful: Advanced Quantum Field Theory; Symmetries, Fields and Particles; Standard Model

References

[1] H. Rothe, *Lattice Gauge Theories: An Introduction*, World Scientific, 1992.

[2] I. Montvay & G. Münster, *Quantum fields on a lattice*, Cambridge University Press, 1994.

[3] T. Degrand & C. DeTar, *Lattice Methods for Quantum Chromodynamics*, World Scientific, 2006.

[4] C. Gattringer & C. Lang, *Quantum Chromodynamics on the Lattice: An Introductory Presentation*, Springer, 2010.

[5] M. R. Shepherd, J. J. Dudek and R. E. Mitchell, *Searching for the rules that govern hadron construction*, Nature **534**, 487 (2016) [arXiv:1802.08131].

[6] M. R. Pennington, *Evolving images of the proton: Hadron physics over the past 40 years*, J. Phys. G **43**, 054001 (2016) [arXiv:1604.01441] (an overview of hadron physics and puts this topic in a broader context).

[7] Useful papers, including reviews, can be found online on the arXiv (http://arxiv.org/).

92. Chiral Fermions on the Lattice .... Professor D. Tong

No one knows how to write down a discrete version of the laws of physics. The problem is that the Standard Model is a "chiral gauge theory", meaning that the left-handed and right-handed fermions experience different forces. There are topological reasons, enshrined in the Nielsen-Ninomiya theorem, which mean that it is difficult to construct discrete (or "lattice") versions of such theories. The purpose of this essay is to explain this problem and to describe attempts to circumvent it.

Relevant Courses

Essential: Quantum Field Theory, Advanced Quantum Field Theory, Standard Model

References

[1] An introduction to lattice fermions can be found in chapter 4 of the lectures: http://www.damtp.cam.ac.uk/user/tong/gaugetheory.html

[2] Lectures on chiral lattice fermions, focussing on the domain wall and overlap approaches, can be found in:

M. Lüscher, "Chiral Gauge Theories Revisited", hep-th/0102028

D. Kaplan, "Chiral Symmetry and Lattice Fermions", arXiv:0912.2560

93. Strichartz Estimates and Nonlinear Schrödinger Equations .... Dr C.M. Warnick

Nonlinear Schrödinger equations arise as models for various physical phenomena including Bose-Einstein condensates, light propagation in optical fibres, waves on deep inviscid water and Langmuir waves in hot plasmas.

Strichartz estimates are estimates for the linear Schrödinger equation which control *spacetime* $L^p_t L^q_x$ norms in terms of the data. These estimates capture the dispersive behaviour of solutions to the linear Schrödinger equation, and are of considerable importance in the study of the non-linear problem. The first such estimate was established in [1], but since then many refinements have been established, see for example [2, 3] and references therein.

The purpose of this essay is to investigate Strichartz estimates for the Schrödinger operator and applications to semilinear Schrödinger equations. As a typical application, the local well-posedness [2] may be considered, but a more ambitious option would be to look at results concerning global well posedness and scattering for the power law nonlinearity in the subcritical [4] or critical case [5].

Relevant Courses

Essential: Analysis of PDE

Useful: Distribution Theory and Applications

References

[1] R Strichartz *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations.* Duke Math. J. **44** (1977), no. 3, 705–714. doi:10.1215/S0012-7094-77-04430-1

[2] Kato, T. *On nonlinear Schrödinger equations.* Annales de l'I.H.P. Physique théorique 46.1 (1987): 113-129. http://eudml.org/doc/76348.

[3] M Keel, T Terence *Endpoint Strichartz estimates* Am. J. Math., Volume **120** (1998), No. 5, pp. 955-980

[4] Colliander, J., Keel, M., Staffilani, G., Takaoka, H., & Tao, T. (2004). *Global existence and scattering for rough solutions of a nonlinear Schrödinger equation on* $\mathbb{R}^3$. Communications on pure and applied mathematics, 57(8), 987-1014

[5] Colliander, J., Keel, M., Staffilani, G., Takaoka, H., & Tao, T. *Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in* $\mathbb{R}^3$. Annals of Mathematics (2008): 767-865.

94. Linear Fields on Black Hole Backgrounds .... Dr C.M. Warnick

The first step to understanding black hole stability is to study in detail the behaviour of linear fields on a fixed black hole background. In recent years, our understanding of this problem has advanced substantially. Through the vector field method and its refinements, considerable progress has been made to understand solutions of the main model equations: the wave equation, Maxwell's equation and the linearised gravitational equations. A robust understanding has been obtained of the decay properties of fields in the black hole exterior for a variety of black hole solutions to Einstein's equations, with and without cosmological constant. There has also been significant work to understand the behaviour of linear fields in the black hole interior, with implications for the strong cosmic censorship conjecture.

A good essay will survey a selection of recent results, and treat at least one result in detail.

Relevant Courses

Essential: General Relativity, Black Holes

Useful: Analysis of PDE

References

[1] M. Dafermos and I. Rodnianski, "Lectures on black holes and linear waves," Clay Math. Proc. **17** (2013) 97 [arXiv:0811.0354 [gr-qc]].

[2] M. Dafermos, I. Rodnianski and Y. Shlapentokh-Rothman, "Decay for solutions of the wave equation on Kerr exterior spacetimes III: The full subextremal case $|a| < M$," arXiv:1402.7034 [gr-qc].

[3] C. M. Warnick, "On quasinormal modes of asymptotically anti-de Sitter black holes," Commun. Math. Phys. **333** (2015) no.2, 959, doi:10.1007/s00220-014-2171-1 [arXiv:1306.5760 [gr-qc]].

[4] S. Aretakis, "Horizon Instability of Extremal Black Holes," Adv. Theor. Math. Phys. **19** (2015) 507, doi:10.4310/ATMP.2015.v19.n3.a1 [arXiv:1206.6598 [gr-qc]].

95. Stationary Spacetimes and Finsler Geometry .... Dr C. M. Warnick

For any spacetime, it is important to try and understand the behaviour of light rays. In a static spacetime, the problem of finding the null geodesics can be reduced by a conformal transformation to studying the geodesics of a related Riemannian metric, which defines the *optical geometry* of the spacetime. For a stationary spacetime, the optical geometry is not Riemannian, but rather defined by a type of Finsler metric.

Finsler geometry generalises Riemannian geometry by not requiring the line element to be defined by a quadratic form. It arises naturally in certain variational problems. A good attempt should include the basic definitions and results from Finsler geometry, the connection to stationary spacetimes as well as some further applications.

Relevant Courses

Essential: General Relativity

Useful: Black Holes, Differential Geometry

References

[1] G. W. Gibbons, C. A. R. Herdeiro, C. M. Warnick and M. C. Werner, "Stationary Metrics and Optical Zermelo-Randers-Finsler Geometry," Phys. Rev. D **79** (2009) 044022, doi:10.1103/PhysRevD.79.044022 [arXiv:0811.2877 [gr-qc]].

[2] G. W. Gibbons and C. M. Warnick, "Traffic Noise and the Hyperbolic Plane," Annals Phys. **325** (2010) 909, doi:10.1016/j.aop.2009.12.007 [arXiv:0911.1926 [gr-qc]].

[3] G. W. Gibbons, "A Spacetime Geometry picture of Forest Fire Spreading and of Quantum Navigation," arXiv:1708.02777 [gr-qc].

96. Markov Chain Monte Carlo for Tall Data .... Dr S. A. Bacallado

Markov chain Monte Carlo methods can be credited with the popularity of Bayesian inference since their development in the 1990s, but they remain computationally expensive in many applications. Standard algorithms do not scale well with *tall* data, that is, when there are large numbers of conditionally independent observations, as they require computing the full likelihood at each iteration.

This essay is meant to review a set of scalable algorithms for Bayesian inference developed in the last 4 years, starting from the review of Bardenet et al. [1], and focusing on a few of the methods discussed in [2-6] or other references in the review. Those so inclined can implement a subset of the methods and present a numerical experiment or practical application.

Relevant Courses

Essential: Bayesian Modelling and Computation

References

[1] Bardenet, R., Doucet, A., and Holmes, C. (2017), "On Markov Chain Monte Carlo Methods for Tall Data", *Journal of Machine Learning Research*, 18, pp. 1–43.

[2] Bardenet, R., Doucet, A., and Holmes, C. (2014), "Towards Scaling up Markov Chain Monte Carlo: An Adaptive Subsampling Approach", in *Proceedings of The 31st International Conference on Machine Learning*, pp. 405–413.

[3] Alquier, P., Friel, N., Everitt, R., and Boland, A. (2016). "Noisy Monte Carlo: Convergence of Markov chains with approximate transition kernels". *Statistics and Computing*, 26(1-2), p. 29–47.

[4] Korattikara, A., Chen, Y., and Welling, M. (2014), "Austerity in MCMC Land: Cutting the Metropolis-Hastings Budget", in *Proceedings of the 31st International Conference on Machine Learning*, pp. 181–189.

[5] MacLaurin, D., and Adams, R. P. (2014), "Firefly Monte Carlo: Exact MCMC With Subsets of Data," in *Proceedings of the 30th Conference on Uncertainty in Artificial Intelligence*

[6] Minsker, S., Srivastava, S., Lin, L., and Dunson, D. (2014), "Scalable and Robust Bayesian Inference via the Median Posterior," in *Proceedings of the 31st International Conference on Machine Learning.*

97. Chaining and Metric Entropy .... Professor R. Nickl

Many mathematical challenges in contemporary high-dimensional and non-parametric statistics involve the control of the stochastic size of suprema of collections of random variables, that is, of expressions of the form $E \sup_{t \in T} X_t$, where $(X_t : t \in T)$ is a stochastic process indexed by the some general index set T. For centred *Gaussian* processes this question can be answered entirely in terms of the complexity of the metric space (T, d), where $d^2(s,t) = E(X_s - X_t)^2, s, t \in T$. For other stochastic processes, such as empirical or Rademacher processes, characterising the size of such suprema can be a more delicate problem. For *sub-Gaussian processes* a universal tool that often provides statistically useful bounds is *Dudley's metric entropy bound* based on *chaining* and on Kolmogorov's notion of the *d-metric entropy* of the set T. More sophisticated techniques, such as 'generic chaining', are sometimes required to sharpen such results.

The purpose of this essay is to summarise some main ideas in this area in a coherent way, and to try to explain some of the proofs, as well as potential applications to high-dimensional and non-parametric statistics. References that include the key material are [3, 4] and applications are presented in [2,4,5]. A more ambitious student can also look into some exciting recent work on the problem [1,6,7].

Relevant Courses

Some background in analysis, statistics and probability is helpful.

References

[1] W. Bednorz, R. Latala, On the boundedness of Bernoulli processes. Ann. of Math. (2) 180 (2014), 11671203

[2] P. Bhlmann, S. van de Geer, *Statistics for high-dimensional data.* Springer, Heidelberg, 2011.

[3] R. M. Dudley, *Uniform central limit theorems*, 2nd edition, Cambridge University Press (2014).

[4] E. Giné, R. Nickl, *Mathematical foundations of infinite-dimensional statistical models.* Cambridge University Press, Cambridge (2016).

[5] V. Koltchinskii, *Oracle inequalities in empirical risk minimization and sparse recovery problems.* Lecture Notes in Mathematics, 2033, Springer, Heidelberg, 2011

[6] M. Talagrand, *Upper and lower bounds for stochastic processes*, Springer, Heidelberg, 2014

[7] R. van Handel, Chaining, interpolation, and convexity. J. Eur. Math. Soc., 20 (2018), 24132435

98. Instantons ....

Professor N. Dorey

Instantons are finite action classical solutions of the equations of motion of a field theory in Euclidean spacetime [1,2]. At weak coupling, they appear as saddle points of the path integral and yield exponentially small contributions to observables. However, in the presence of fermions, they can contribute to quantities which are not corrected at any finite order in perturbation theory. This is particularly true in supersymmetric theories where they sometimes yield the exact results for special protected observables (see [3,4] for a review). Instanton solutions in four dimensional gauge theory are also of interest to mathematicians; thanks to the ADHM construction [5], a general solution is available on $\mathbb{R}^4$ and, on more general four-manifolds, the moduli space of instanton solutions is the starting point for constructing topological invariants known as Donaldson invariants (see eg [6]). Instanton effects are also important in string theory where they play a key role in the phenomenon of mirror symmetry [7]. The essay should start with the basics of instanton physics, but can progress to survey one or more advanced topics.

Relevant Courses

Essential: Quantum Field Theory, Advanced Quantum Field Theory

Useful: Solitons, Supersymmetry, String Theory

References

[1] A. A. Belavin, A. M. Polyakov, A. S. Schwartz and Y. S. Tyupkin, "Pseudoparticle Solutions of the Yang-Mills Equations," Phys. Lett. B **59** (1975) 85 [Phys. Lett. **59B** (1975) 85]. [2] G. 't Hooft, "Computation of the Quantum Effects Due to a Four-Dimensional Pseudoparticle," Phys. Rev. D **14** (1976) 3432

[3] David Tong, "TASI lectures on Solitons", Lecture 1
http://www.damtp.cam.ac.uk/user/tong/tasi/instanton.pdf

[4] S. Vandoren and P. van Nieuwenhuizen, "Lectures on instantons," arXiv:0802.1862 [hep-th].

[5] M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld and Y. I. Manin, Phys. Lett. A **65** (1978) 185.
[6] D. Freed and K. Uhlenbeck. "Instantons and four-manifolds" (Springer 1991)

[7] D. R. Morrison and M. R. Plesser, "Summing the instantons: Quantum cohomology and mirror symmetry in toric varieties," Nucl. Phys. B **440** (1995) 279

99. Statistical Inference for Discretely Observed Compound Poisson Processes and Related Jump Processes ....

Dr A. J. Coca

Stochastic processes are used in innumerable applications to model random dynamical systems. In many of these applications random shocks take place and a class of particular importance is that of jump processes; prominent examples arise from seismology, neuroscience, finance, queues, telecommunication networks, radiation detection, and many more. The data acquisition systems of most of these only register values of the underlying jump process at discrete times and statistical inference for the process becomes a statistical (generally nonlinear) inverse problem. This is an active area of research and, in particular, nonparametric inference for discretely observed compound Poisson processes (CPPs) has received much attention since the seminal work in [1]. The importance of CPPs is two-fold: they are one of the "off-the-shelf" processes first used in practice to model jumps; and, from a more theoretical perspective, they sit at the intersection of fundamental classes of jump processes such as Lévy processes, renewal processes, Poisson point processes, etc. whilst they still retain much of the mathematical structure and challenges of each class constituting a tractable but representative model to work with.

I envisage considerable flexibility in this project: for a student wishing to acquire a general overview, they could review the literature on non-Bayesian estimation of CPPs (see [1,2,3,4,6]), potentially connecting it to respective literature on more general Lévy or renewal processes; for a more technical project, they could review some of the literature in more detail focusing on a concept from nonparametric statistics such as adaptive estimation (see [3,4]), uncertainty quantification (see [1,2]), Bayesian inference (see [7,8]) or information bounds (see [5,9]); and, those with a more computational taste could implement and compare existing methodologies, including spectral ([2,6]), Bayesian ([7,8]), data-augmentation ([7]) and model selection techniques ([3,4]), and others less studied such as the bootstrap and optimisation methods. More references are available and there is room for some novel work in all these directions.

Relevant Courses

Essential: None

Useful: Topics in Statistical Theory and Advanced Probability.

References

[1] Buchmann, B. and Grübel, R.: Decompounding: an estimation problem for Poisson random sums. Ann. Statist. **31**(4), 1054–1074 (2003)

[2] Coca, A.J.: Efficient nonparametric inference for discretely observed compound Poisson processes. Probab. Theory Related Fields, **170**(1-2), 475–523 (2018)

[3] Coca, A.J.: Adaptive nonparametric estimation for compound Poisson processes robust to the discrete-observation scheme. arXiv preprint, arXiv:1803.09849 (2018)

[4] Duval, C.: Density estimation for compound Poisson processes from discrete data. Stochastic Process. Appl. **123**(11), 3963–3986 (2013)

[5] Duval, C. and Hoffmann, M.: Statistical inference across time scales. Electronic Journal of Statistics, **5**, 2004–2030 (2011)

[6] van Es, B., Gugushvili, S. and Spreij, P.: A kernel type nonparametric density estimator for decompounding. Bernoulli, **13**(3), 672–694 (2007)

[7] Gugushvili, S., van der Meulen, F. and Spreij, P.: A non-parametric Bayesian approach to decompounding from high frequency data. Stat. Inference Stoch. Process. **21**(1), 53–79 (2018)

[8] Nickl, R. and Söhl, J.: Bernstein - von Mises theorems for statistical inverse problems II: compound Poisson processes. arXiv preprint, arXiv:1709.07752 (2017)

[9] Trabs, M.: Information bounds for inverse problems with application to deconvolution and Lévy models. Ann. l. H. Poincare-Pr. **51**(4), 1620–1650 (2015)

100. On the structure of Chevalley groups over local fields .... Dr B. L. Romano

The main goal of this essay is to read and understand the paper "On some Bruhat decomposition and the structure of Hecke rings of p-adic Chevalley groups" by Iwahori and Matsumoto. To do so, you'll first learn about the structure of Chevalley groups (e.g. root groups, commutator relations, (B, N)-pairs) by reading, e.g. Carter's *Simple groups of Lie type* (Steinberg's notes are also a good reference). You'll also need a basic understanding of local fields: students who have not taken a course covering this topic can find details in, e.g., Serre's *Local fields*. Your essay should discuss the concepts in Iwahori–Matsumoto (e.g. affine Weyl groups and root systems, maximal compact subgroups) via explicit examples. Interested students should also gain some context for the importance of Iwahori–Matsumoto's results by reading about the representation theory of p-adic groups: Kim's "Supercuspidal representations: construction and exhaustion" is a nice survey paper for this, and Chapter 1 of Bushnell–Henniart's *The local Langlands conjecture for* GL(2) is good background material.

Relevant Courses

Essential: Lie algebras and their representations

Useful: Algebraic number theory (or any course that provides background in local fields)

References

[1] Iwahori, N. and Matsumoto, H. *On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups*, Inst. Hautes Études Sci. Publ. Math. 25, 1965.

[2] Carter, Roger W. *Simple groups of Lie type*, Wiley, New York, 1989.

[3] Serre, Jean-Pierre, *Corps locaux*, Hermann, Paris, 1968.

[4] Kim, Ju-Lee, *Supercuspidal representations: construction and exhaustion*, Ottawa lectures on admissible representations of reductive p-adic groups, Fields Inst. Monogr. 26, 2009.

[5] Bushnell, Colin J. and Henniart, Guy, *The local Langlands conjecture for* GL(2), Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] 335, Springer-Verlag, Berlin, 2006.

101. Kodaira's problem .... Dr R. Svaldi

Kähler manifolds are the main class of objects that are studied in complex geometry because of their rich structure. A large source of examples of Kähler manifolds is given by projective ones, i.e., those that can be embedded holomorphically in some projective space. It is natural to ask exactly how large the class of smooth projective varieties is within that of compact Kähler manifolds. Kodaira showed that there is a cohomological criterion to determine when a compact Kähler manifold is projective. But projectivity is not stable under deformations of the complex structure.

In the course of his analysis of compact Kähler surfaces, Kodaira, [2], proved that every such surface S possesses deformations that become projective. This result was later generalized by Buchdahl, [1].

Kodaira's problem asks whether this phenomenon occurs in (complex) dimension higher than 2. Voisin, [4], has constructed a series of examples in dimension 4 and higher for which Kodaira's problem has a negative answer. She later proved that even the birational version of the problem – that is, if a birational model of a compact Kähler manifold can be deformed to a projective variety – has also negative answer, see [5].

In this essay, you will present Voisin's constructions, after introducing Kodaira's problem in the framework of Kähler geometry.

Relevant Courses

Essential: Part III Algebraic Geometry, Algebraic Topology, and Complex Manifolds.

Useful: Part III Algebra.

References

[1] Buchdahl, N. *Algebraic deformations of compact Khler surfaces II.* Math. Z., 258:493–498, 2008.

[2] K. Kodaira. *On compact analytic surfaces. II, III.* Ann. of Math. (2) 77 (1963), 563–626 ; ibid., 78 :1–40, 1963.

[3] C. Voisin. *Hodge Theory and Complex Algebraic Geometry I, II*, volume 76-77 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2003.

[4] C. Voisin. *On the homotopy types of compact Kähler and complex projective manifolds.* Invent. Math., 157:329–343, 2004.

[5] C. Voisin. *On the homotopy types of Khler manifolds and the birational Kodaira problem.* J. Differential Geom., 72(1):43–71, 2006.

102. Semipositivity properties of the tangent bundle ....

Dr R. Svaldi

In the study of algebraic varieties, a huge role is played by the positivity properties of the tangent bundle of an algebraic variety. The word positivity here refers to the existence of metrics with positive curvature on some portion either of the tangent bundle or of its tensor/exterior powers. For example, if at the general point of a smooth projective variety there are curves along which the determinant of the tangent bundle has positive curvature then the variety itself is uniruled, that is, is covered by rational curves, see [5]. Actually, one of the central conjectures in the classification of algebraic varieties, classically attributed to Mumford, predicts that the same conclusion should hold under a much weaker condition: namely, when the determinant of the cotangent bundle and its tensor powers have no sections.

The aim of this essay is to explore some of the fundamental theorems and techniques that describe the properties of the tangent bundle in relation to its positivity, or lack thereof.

One of the first results in this area – that should constitute the core of the essay – is Miyaoka's Semipositivity Theorem, [4, 5]. In order to give a proof of this theorem, you will have to familiarize yourself with a plethora of techniques and results that are at the foundations of modern birational geometry, see [1], for example.

This is a rather advanced essay, hence, ideally, you would already be comfortable using tools from algebraic geometry at the level of chapter 2-3 of Hartshorne and beyond.

Relevant Courses

Essential: Part III Algebraic Geometry, and Algebra.

Useful: Part III Complex Manifolds, and Algebraic Topology.

References

[1] Kollár, J. *Rational curves on algebraic varieties.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 32. Springer-Verlag, Berlin, 1996. viii+320 pp.

[2] Kollár, J. et al. *Flips and abundance for algebraic threefolds.* A summer seminar at the University of Utah (Salt Lake City, 1991). Astérisque 211 (1992), 272 pages

[3] Miyaoka, Y. *Deformations of a morphism along a foliation and applications.* Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), 245268, Proc. Sympos. Pure Math., 46, Part 1, Amer. Math. Soc., Providence, RI, 1987.

[4] Miyaoka, Y. *The Chern classes and Kodaira dimension of a minimal variety.* Algebraic geometry, Sendai, 1985, 449–476, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam, 1987.

[5] Miyaoka, Y., Mori, S. *A numerical criterion for uniruledness.* Ann. of Math. (2) 124 (1986), no. 1, 65–69.

103. Introduction to the Minimal Model Programme ....

Dr R. Svaldi

In algebraic geometry, one of the main goals is to classify algebraic varieties. Among the many possible approaches to this problem, one can choose to consider two varieties to be equivalent when they are birational, that is, when the fields of rational functions are isomorphic. The idea is then that, within a given equivalence class of birational varieties, one should identify a representative whose geometric structure is simple to analyze and to break down into simpler pieces. Such models are conjectured to exist and their construction is the aim of the so-called Minimal Model Programme (MMP in short), initiated by Mori in the late 1970's and then developed by many authors in the course of the past 40 years.

In this essay, you will look at some of the ideas and the techniques behind the MMP.

You will first introduce the notion of log pair and of singularities of pairs. This is a fundamental tool nowadays in the study of the birational structure of algebraic varieties, as you can read in the very beautiful and modern [2]. The final goal is to explain how the algorithm for the birational classification of varieties works, what steps constitute such algorithm and you will provide a detailed description of the the main ingredients involved and a proof of some of the main results, see [1, 3] – to be determined with me.

Relevant Courses

Essential: Part III Algebraic Geometry, and Algebra.

Useful: Part III Complex Manifolds.

References

[1] Debarre, O. *Higher-dimensional algebraic geometry.* Universitext. Springer-Verlag, New York, 2001. xiv+233 pp.

[2] Kollár, J. *Singularities of the minimal model program.* With a collaboration of Sándor Kovács. Cambridge Tracts in Mathematics, 200. Cambridge University Press, Cambridge, 2013. x+370 pp.

[3] Kollár, J., Mori, S. *Birational geometry of algebraic varieties.* With the collaboration of C. H. Clemens and A. Corti. Translated from the 1998 Japanese original. Cambridge Tracts in Mathematics, 134. Cambridge University Press, Cambridge, 1998. viii+254 pp.

[4] Lazarsfeld, R. *Positivity in algebraic geometry. I. Classical setting: line bundles and linear series.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 48. Springer-Verlag, Berlin, 2004. xviii+387 pp.

[5] Lazarsfeld, R. *Positivity in algebraic geometry. II. Positivity for vector bundles, and multiplier ideals.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 49. Springer-Verlag, Berlin, 2004. xviii+385 pp.

104. Pursuit on Graphs ....

Professor I. Leader

There has recently been much interest in pursuit questions on graphs. Typically, we have a number of pursuers, working as a team to catch an evader. If this takes place on a graph (so the players live on the vertices of the graph, and in each time-step they move to an adjacent vertex), how many pursuers are needed? And how does this relate to properties of the graph? Most work centres around the conjecture of Meyniel, still unproved, that the number of pursuers need be no more than about the square-root of the number of vertices.

The essay would focus on some results for general graphs, and also on specific cases of interest like random graphs.

Relevant Courses

Essential: None

Useful: Combinatorics

References

[1] Cops and robbers in graphs with large girth and Cayley graphs, P.Frankl

[2] A new bound for the cops and robbers problem, A.Scott and B.Sudakov

[3] Chasing robbers on random graphs: zigzag theorem, T.Luczak and P.Pralat

[4] On a generalization of Meyniel's conjecture on the cops and robbers game, N.Alon and A.Mehrabian

105. Hamiltonian Cycles and Spheres in Hypergraphs ....

Professor I. Leader

The notion of a Hamilton cycle (a cycle through all the vertices) in a graph also makes sense for a hypergraph. There are various versions. One is that we may list the vertices cyclically in such a way that every interval of length k (where the hypergraph consists of k-sets) belongs to the hypergraph. Is there an analogue of the well-known Dirac theorem for graphs, which states that if a graph has minimum degree at least $n/2$ then it has a Hamilton cycle?

The aim of the essay is to focus on some classical results on this question, and also on some very recent work on 'Hamilton spheres'.

Relevant Courses

Essential: None

Useful: Combinatorics

References

[1] An approximate Dirac-type theorem for k-uniform hypergraphs, V.Rodl, A.Rucinski and E.Szemeredi, Combinatorica volume 28 (2008), 229-260.

[2] Spanning surfaces in 3-graphs, A.Georgakopoulos, J.Haslegrave, R.Montgomery and B.Narayanan, available at arXiv:1808.06864

106. Gravity currents passing over cavities .... Professor S. B. Dalziel

High-Reynolds-number gravity currents are found throughout the natural and man-made environments [1] and an understanding of gravity currents provides a good starting point to the nearly identical phenomenon of 'intrusions' between two layers of different density. At sufficiently large Reynolds numbers, both gravity currents and intrusions are capable of entraining ambient fluid due to turbulent mixing.

This essay will explore a dense gravity current propagating across a boundary beneath an ambient fluid of uniform density, but for the case where the boundary is interrupted by an isolated cavity filled with a fluid of density greater than that of the ambient. Shear across such a cavity is known to drive mixing and entrainment [2]. This situation is relevant for flows such as a gravity current flowing across the floor of the ocean encountering a depression containing fluid with a greater density, or the urban environment where cold air (from radiative night time cooling) or dense gas is trapped between buildings prior to a cold front crossing the urban environment [3,4].

An essay on this topic will begin with an overview of gravity currents and intrusions before undertaking a review of existing literature relating to this specific problem. From here, the essay could proceed in one (or both) of two directions to assess the impact, propagation and/or mixing associated with the presence of the cavity. Either a set of simple laboratory experiments could be conducted, or shallow water theory could be adapted to provide insight into the phenomenon.

Relevant Courses

Essential: None

Useful: Fluid Dynamics of the Environment

References

[1] Simpson, J.E. 1997 *Gravity currents in the environment and the laboratory*, 2nd Edn. Cambridge University Press.

[2] Strang, E.J. and Fernando, H.J.S., 2001. Entrainment and mixing in stratified shear flows. *Journal of Fluid Mechanics*, **428**, pp.349-386.

[3] Kaye, N.B. and Baratian-Ghorghi, Z., 2017. Role of ambient turbulence in dense gas dispersion from confined urban regions. *Journal of Hydraulic Engineering*, **144**(2), p.06017029.

[4] Kirkpatrick, M.P., Armfield, S.W. and Williamson, N., 2012. Shear driven purging of negatively buoyant fluid from trapezoidal depressions and cavities. *Physics of Fluids*, **24**(2), p.025106.

107. Clifford algebras and their connection to elementary particle physics .. Dr C. Furey

It is no secret that the Clifford algebra $Cl(1,3)$ underlies Dirac's famous equation. However, this does not mark the end of the known connection between Clifford algebras and elementary particle physics. The Clifford algebra $\mathbb{C}l(10)$ can furthermore be seen to underlie Spin(10) and SU(5) grand unified theories, in addition to the Pati-Salam model. This essay explores a multitude of ways in which Clifford algebras have been found to be inextricably cemented into our familiar theories of fundamental physics.

Relevant Courses

Essential: Quantum Field Theory

Useful: Symmetries, Fields and Particles; Standard Model

References

[1] J. Baez, J. Huerta, *The algebra of grand unified theories,* https://arxiv.org/abs/0904.1556

[2] R. Ablamowicz, *Construction of spinors via Witt decomposition and primitive idempotents: a review,* Clifford algebras and spinor structures. Kluwer Academic Publishers, 1995.

[3] A. Barducci, F. Buccella, R. Casalbuoni, L. Lusanna, and E. Sorace, *Quantized grassmann variables and unifed theories,* Phys. Letters B, 1977

[4] A. Conway, *Quaternion treatment of relativistic wave equation,* Proceedings of the Royal Society of London, Series A, Mathematical and physical sciences, 162, No 909 (1937)

108. Modelling Quantum Dynamics Using Random Unitary Circuits .... Dr. A. Lamacraft

Quantum entanglement, a physical phenomenon in which distinct degrees of freedom are correlated and the quantum state of one particle cannot be written independently of the others, plays an important role in topics such as many-body localization and holography. Recent work has shown that random unitary circuits (RUCs) provide a minimally structured, discrete-time model for entanglement growth in many-body systems [1]. It has been shown analytically and computationally that in one and higher spatial directions, the exponents of operator growth in these systems match those of the Kardar-Parisi-Zhang universality class [2]. This essay will discuss the motivation behind this work, the mathematical background of random unitary circuits, and one or more of the following topics: operator spreading and ways to quantify this (out-of-time-order correlator), the importance of conservation laws in these systems, continuous time analogues to the RUC model [3], and comparison to other models of entanglement growth involving fractons [4] and the SYK model [5].

Relevant Courses

Essential: None

Useful: Quantum Computation, Statistical Field Theory

References

[1] Nahum, A., Ruhman, J., Vijay, S., & Haah, J. (2017). Quantum Entanglement Growth under Random Unitary Dynamics. Physical Review X, 7(3), 031016. https://doi.org/10.1103/PhysRevX.7.031016

[2] Nahum, A., Vijay, S., & Haah, J. (2018). Operator Spreading in Random Unitary Circuits. Physical Review X, 8(2), 021014. https://doi.org/10.1103/PhysRevX.8.021014

[3] Rowlands, D. A., & Lamacraft, A. (2018). Noisy Coupled Qubits: Operator Spreading and the Fredrickson-Andersen Model. Retrieved from http://arxiv.org/abs/1806.01723

[4] Pai, S., Pretko, M., & Nandkishore, R. M. (2018). Localization in fractonic random circuits, 115. Retrieved from http://arxiv.org/abs/1807.09776

[5] Roberts, D. A., Stanford, D., & Streicher, A. (2018). Operator growth in the SYK model. Journal of High Energy Physics, 2018(6), 122. https://doi.org/10.1007/JHEP06(2018)122

109. Sheaves on Locales and Internal Locales ....

Professor P. T. Johnstone

The aim of an essay on this topic would be to develop the theory of sheaves on a locale (or ‘point-free space’) and to describe the notion of internal locale in such a category, with the goal of establishing the equivalence between internal locales in the category of sheaves on X and locales over X. For background material on locales, see [1] and [2]; for internal locales, see [3], [4] and [5].

Relevant Courses

Essential: Category Theory

References

[1] P.T. Johnstone, *Stone Spaces*, Cambridge Studies in Advanced Mathematics 3 (C.U.P., 1982).

[2] S.J. Vickers, *Topology via Logic*, Cambridge Tracts in Theoretical Computer Science 5 (C.U.P., 1989)

[3] M.P. Fourman and D.S. Scott, Sheaves and logic, in *Applications of Sheaves*, Lecture Notes in Mathematics 753 (Springer, 1979), 302–401.

[4] A. Joyal and M. Tierney, *An Extension of the Galois Theory of Grothendieck*, Memoirs Amer. Math. Soc. 309 (1984).

[5] P.T. Johnstone, *Sketches of an Elephant: a Topos Theory Compendium*, chapter C1, Oxford Logic Guides 44 (O.U.P., 2002), 861–889.

110. Conformal Field Theories . . . .
Professor H. Osborn

Conformal Quantum Field Theories in more than two dimensions have become a subject of intense interest in the last ten years. They arise as fixed points at large distances and are relevant in understanding quantum field theories in four dimensions as well as in three for applications in condensed matter physics. CFTs have no mass scales and are expressed in terms of a spectrum of operators with various scaling dimensions and spins. The bootstrap programme, which depends on basic quantum field theory properties such unitarity and crossing symmetry, provide strong constraints on the possible operators subject to the assumed symmetries of the theory. The numerical bootstrap provides very accurate results for the critical exponents, which depend on the scaling dimensions of various operators, in the three dimensional Ising model. Although much work is numerical there are many analytical results.

The essay should describe the basic framework of CFTs and the motivations for interest in them. It should discuss the essential ideas behind the bootstrap programme. It might consider more specialised topics such as conformal blocks or possibly conformal perturbation theory. Various reviews are contained in [1, 2, 3]. The last is particularly compendious and it is not necessary to digest all of it. A useful background, which predates the recent surge of interest in CFTs, is contained in [4].

References

[1] S. Rychkov, *EPFL Lectures on Conformal Field Theory in $D \geq 3$ Dimensions*. SpringerBriefs in Physics. 2016. `arXiv: 1601.05000 [hep-th]`.

[2] D. Simmons-Duffin, "The Conformal Bootstrap" in *Proceedings, Theoretical Advanced Stufy Institute in Elementary Particle Physics: New Frontiers in Fields and Strings (TASI 2015): Boulder, CO, USA, June 1-26, 2015*, pp. 1-74, 2017. `arXiv:1602.07982 [hep-th]`.

[3] D. Poland, S. Rychkov and A. Vichi, "The Confomal Bootstrap: Theory, Numerical Techniques and Applications" `arXiv:1805.04405 [hep-th]`.

[4] J. L. Cardy, *Scaling and renormalization in statistical physics*. Cambridge lecture notes in physics: 3. Cambridge, UK: Univ. Pr., 1996. 238 p.

111. Stochastic Graphical Games . . . .
Dr M. Elliott

Stochastic games are a generalisation of Markov decision processes. The aim of the essay would be to understand their equilibria, in the sense of Nash, especially for graphical games. This involves some intricate and fascinating mathematics. The topics to be covered are: Stochastic games as a generalisation of Markov decision processes and repeated games; stochastic graphical games, their properties and computation of the Nash equilibrium; the asymptotic behaviour of these games with respect to their equilibrium points.

Relevant Courses

Essential: None

Useful: Advanced Probability

References

These can all be found online, or are available from Dr Elliott.

[1] 'Markov decision processes and stochastic games with total effective payoff', by Boros, Elbassioni, Gurvich and Makino

[2] 'On Nath equilibria in stochastic games', by Vhatterjee, Jurdzinski and Majumdar

[3] 'Stochastic games for N players', by Bensoussan and Frehse

112. Stretch factors of pseudo-Anosovs ....

Dr R. C. H. Webb

The *mapping class group* Mod(S) of a surface S is the group of homeomorphisms $S \to S$ modulo isotopy. The mapping class group occurs naturally in several different fields e.g. as the (orbifold) fundamental group of the moduli space of Riemann surfaces, and, in the study of fibre bundles with fibre S.

An important landmark theorem is the Nielsen–Thurston Classification proved by W. Thurston in the 70s. It states that an element $f \in \text{Mod}(S)$ of infinite order must either preserve a collection of non-trivial isotopy classes of simple closed curves on S (reducible), or, has a representative homeomorphism which (outside of finitely many singular points) stretches S by $\lambda > 1$ in one direction and contracts by $1/\lambda$ in another direction (pseudo-Anosov). The number λ is called the *stretch factor* or *dilatation* and is a conjugacy invariant of f. Most elements of Mod(S) are pseudo-Anosov, and in our efforts to study them we run into questions in geometry, dynamics, and number theory.

The essay should introduce the notion of a pseudo-Anosov f, give an overview of a proof of the Nielsen–Thurston classification, and explain why the stretch factor $\lambda = \lambda(f)$ is an algebraic integer. In fact, Fried showed that λ is a bi-Perron algebraic unit. It is a difficult open problem to determine whether every bi-Perron algebraic unit is the stretch factor of some pseudo-Anosov.

The essay can then finish on one of two different topics. The first choice would be to explain some recent results on the stretch factors of pseudo-Anosovs e.g. each Salem number has a power that is the stretch factor of some pseudo-Anosov. The second choice would be to explain a theorem of W. Thurston, which states that for any Perron number λ there is some outer automorphism of some free group F_n with stretch factor λ.

Relevant Courses

Essential: Algebraic topology

Useful: Differential geometry, Galois theory

References

Casson, Andrew J., Bleiler, Steven A. Automorphisms of surfaces after Nielsen and Thurston. London Mathematical Society Student Texts, 9. Cambridge University Press, Cambridge, 1988.

Fathi, A., Laudenbach, F., Ponaru, V. Thurston's work on surfaces. Translated from the 1979 French original by Djun M. Kim and Dan Margalit. Mathematical Notes, 48. Princeton University Press, Princeton, NJ, 2012.

Fried, D. Growth rate of surface homeomorphisms and flow equivalence. Ergodic Theory Dynam. Systems 5 (1985), no. 4, 539563.

Pankau, J. Salem number stretch factors and totally real fields arising from Thurston's construction. https://arxiv.org/abs/1711.06374

Thurston, W. Entropy in dimension one. Frontiers in complex dynamics, 339384, Princeton Math. Ser., 51, Princeton Univ. Press, Princeton, NJ, 2014.

Thurston, W. On the geometry and dynamics of diffeomorphisms of surfaces. Bull. Amer. Math. Soc. (N.S.) 19 (1988), no. 2, 417431.

113. The Search for CMB B-mode Polarization from Inflationary Gravitational Waves ....

Dr B. D. Sherwin

Our most promising theory for the early universe involves a phase of cosmic inflation, which not only rapidly expands and flattens the universe, but also generates the primordial density perturbations from quantum fluctuations in the inflaton field. While we have good evidence for inflation, e.g. from the Gaussianity, adiabaticity and near-scale invariance of the scalar density perturbations, one prediction of inflation has not yet been found: many inflationary models produce a stochastic background of primordial gravitational waves. A detection of this background would not only provide a definitive confirmation of inflation, but could also give new insights into the microphysics of inflation and, more broadly, physics at the highest energies.

The best current way of finding this gravitational wave background is to search for a characteristic pattern in the polarization of the Cosmic Microwave Background (CMB), the B-mode polarization. This essay should explain the physics underlying the search for this B-mode polarization pattern, which is currently a major area of research in cosmology.

The essay should first review the calculation of the gravitational wave background produced by standard single-field slow-roll inflation, a standard result described in past Part III lecture notes as well as a comprehensive review of the field (Kamionkowski & Kovetz 2016, henceforth KK16). The essay should also explain why the strength of the gravitational wave background (together with the scalar spectral index) can provide powerful constraints on the properties of inflation, such as the potential shape, energy scale, and field excursion (CMB-S4 2016, KK16).

Drawing on KK16, CMB-S4 2016, past lecture notes and other resources, the essay should provide a (brief) review of the basics of CMB polarization, describe what the CMB B-mode polarization is, and explain why it is a powerful probe of inflationary gravitational waves.

The remaining parts of the essay can, to some extent, be tailored to the student's interests. One option is to explain in detail the major observational challenges in B-mode searches for inflationary gravitational waves, discussing the problems of foregrounds (Bicep/Keck/Planck 2015) and gravitational lensing as well as mitigation methods such as multifrequency cleaning and delensing (Smith et al. 2012). Another option is to focus more on the theoretical background, describing in detail different classes of inflationary models and what these generically predict for B-mode polarization (CMB-S4 2016 and references therein). Students may also discuss a combination of both observational and theoretical aspects.

Relevant Courses

Essential: Cosmology
Useful: Advanced Cosmology, Quantum Field Theory, General Relativity

References

[1] Kamionkowski, M. & Kovetz, E. D. 2016, Annual Review of Astronomy and Astrophysics, 54, 227

[2] CMB-S4 Science Book 2016, arXiv:1610.02743 (mainly chapter 2)

[3] BICEP/Keck/Planck 2015, arXiv:1502.00612, Phys. Rev. Lett. 141 101301

[4] Smith, K. M. et al. 2012, arXiv:1010.0048, JCAP, 06 014

[5] Baumann, D., lecture notes:
http://www.damtp.cam.ac.uk/user/db275/Cosmology/Lectures.pdf

114. Puffs of bubbles in a stratified environment ....
S.B. Dalziel

Much is known about the behaviour of a turbulent buoyant plume driven by a steady isolated source of buoyancy in a stratified environment where the source of buoyancy is due to either heat or a difference in composition. Indeed, the plume model by Morton, Taylor & Turner [1] is one of the most successful simplified models for a turbulent flow and has been shown to work equally well for laboratory scale flows and flows extending over much of the height of the atmosphere. These ideas are readily extended to the case of bubble plumes [2], where small bubbles alter the bulk density of the fluid. However, differences eventually arise due to the 'slip velocity', most noticeably when the rise velocity of the plume becomes comparable with that of an individual bubble. Such bubble plumes occur naturally due to undersea vents and are also utilised in human activities such as the mixing and aeration of stratified lakes [3]. Most previous work, however, has considered a sustained steady source of bubbles.

An essay on this topic will focus on bubble plumes in a stratified environment where the source is generating periodic 'puffs' rather than a continuous release of bubbles. The essay should begin with an overview of previous work on steady bubble plumes and brief description of time-dependent phenomena (e.g. [4]) for either bubble or compositional plumes. From here, the essay could proceed in one (or both) of two directions to report on either a set of laboratory experiments, or the relationship between the sequence of puffs and plume theory could be pursued to make some predictions of the behaviour.

Relevant Courses

Essential: None

Useful: Fluid Dynamics of the Environment

References

[1] Morton, B. R., Taylor, G. I. & Turner, J. S. 1956 Turbulent gravitational convection from maintained and instantaneous sources. *Proc. R. Soc. Lond. A* **234**, 1-32.

[2] McDougall, T.J., 1978. Bubble plumes in stratified environments. *Journal of Fluid Mechanics*, **85**, pp.655-672.

[3] Wüest, A., Brooks, N.H. and Imboden, D.M., 1992. Bubble plume modeling for lake restoration. *Water Resources Research*, **28**, pp.3235-3250.

[4] Craske, J. and van Reeuwijk, M., 2016. Generalised unsteady plume theory. *Journal of Fluid Mechanics*, **792**, pp.1013-1052.

115. Strongly interacting gravity currents ....

S.B. Dalziel

Releases of dense fluid or of light fluid adjacent to a boundary in an otherwise homogeneous environment can lead to the formation of a gravity current [1] that spreads horizontally along the boundary. Such currents occur at high Reynolds number in both the natural and man-made environments. This essay explores if both a dense and light currents, propagating in opposite directions, are released into the same environment. In contrast with the case where both the currents are dense (or both are light) [2], the currents will not collide directly, but unless the environment is very deep, the two currents will still interact, potentially forming a three-layer system.

An essay on this topic will begin with an overview of gravity currents and possibly previous work published on colliding gravity currents. From here, the essay could proceed in one (or both) of two directions to assess the interaction of the currents exploring issues such as their propagation and/or mixing. Either a set of simple laboratory experiments could be conducted, or shallow water theory could be adapted to provide insight into the phenomenon.

Relevant Courses

Essential: None

Useful: Fluid Dynamics of the Environment

References

[1] Simpson, J.E. 1997 *Gravity currents in the environment and the laboratory*, 2nd Edn. Cambridge University Press.

[2] Zhong, Q., Hussain, F., & Fernando, H. (2018). Quantification of turbulent mixing in colliding gravity currents. *Journal of Fluid Mechanics*, **851**, 125-147. doi:10.1017/jfm.2018.488

116. Ultra slow-roll inflation ....

Dr E. Pajer

In the current standard cosmological model, primordial perturbations are generated during a phase of accelerated expansion called inflation. To reproduce the measured (approximate) scale invariance of these perturbations, it is generally assumed that inflation consisted of a phase of quasi de Sitter expansion. This assumption has the very attractive feature that, on superHubble scales, there always exist a solution for which primordial perturbations are constant in time. This is often called the "adiabatic mode" and matches cosmological observations to sub-percent accuracy. There exist however another possibility, namely that inflation was quite different from de Sitter, and nevertheless still produces an approximate scale invariant spectrum of perturbations. This is realized when the Hubble parameter has a small first derivative but a very large second derivative, and is called Ultra slow-roll (USR) inflation. This model is phenomenologically not very appealing but it offers many theoretical challenges to our understanding. In USR inflation, the adiabatic mode is a subleading correction to a growing and non-adiabatic solution. As a consequence all celebrated single-clock consistency conditions (aka soft theorems) fail in this model. In this essay, one first review the USR inflationary background and the calculation of the power spectrum (refs 6, 1 and 2). Second, one studies the calculation of higher n-point functions in inflation (e.g. from 7 and 8) and the related soft theorems (ref 5, and 8-10).

Relevant Courses

Essential: Cosmology, General Relativity, Quantum Field Theory.

Useful: Advanced Cosmology.

References

[1] X. Chen, H. Firouzjahi, M. H. Namjoo and M. Sasaki, "A Single Field Inflation Model with Large Local Non-Gaussianity," EPL **102** (2013) no.5, 59001 doi:10.1209/0295-5075/102/59001 [arXiv:1301.5699 [hep-th]].

[2] M. H. Namjoo, H. Firouzjahi and M. Sasaki, "Violation of Non-Gaussianity Consistency Relation in a Single Field Inflationary Model," Europhys. Lett. **101** (2013) 39001 doi:10.1209/0295-5075/101/39001 [arXiv:1210.3692 [astro-ph.CO]].

[3] B. Finelli, G. Goon, E. Pajer and L. Santoni, "Soft Theorems for Shift-Symmetric Cosmologies," Phys. Rev. D **97** (2018) no.6, 063531 doi:10.1103/PhysRevD.97.063531 [arXiv:1711.03737 [hep-th]].

[4] Y. F. Cai, X. Chen, M. H. Namjoo, M. Sasaki, D. G. Wang and Z. Wang, "Revisiting Non-Gaussianity from Non-Attractor Inflation Models," JCAP **1805** (2018) no.05, 012 doi:10.1088/1475-7516/2018/05/012 [arXiv:1712.09998 [astro-ph.CO]].

[5] K. HinterbiCHLer, L. Hui and J. Khoury, "An Infinite Set of Ward Identities for Adiabatic Modes in Cosmology," JCAP **1401** (2014) 039 doi:10.1088/1475-7516/2014/01/039 [arXiv:1304.5527 [hep-th]].

[6] W. H. Kinney, "Horizon Crossing and Inflation with Large Eta," Phys. Rev. D **72** (2005) 023515 doi:10.1103/PhysRevD.72.023515 [gr-qc/0503017].

[7] E. Lim "Advanced Cosmology : Primordial Non-gaussianities (The asymptotically typo-free version)" lecture notes

[8] J. M. Maldacena, "Non-Gaussian Features of Primordial Fluctuations in Single Field Inflationary Models," JHEP **0305** (2003) 013 doi:10.1088/1126-6708/2003/05/013 [astro-ph/0210603].

[9] B. Finelli, G. Goon, E. Pajer and L. Santoni, "Soft Theorems for Shift-Symmetric Cosmologies," Phys. Rev. D **97** (2018) no.6, 063531 doi:10.1103/PhysRevD.97.063531 [arXiv:1711.03737 [hep-th]].

[10] V. Assassi, D. Baumann and D. Green, "On Soft Limits of Inflationary Correlation Functions," JCAP **1211** (2012) 047 doi:10.1088/1475-7516/2012/11/047 [arXiv:1204.4207 [hep-th]].

117. Bounded gaps between primes .... Dr T. Bloom

One of the oldest problems in mathematics is the Twin Prime Conjecture: that there are infinitely many primes p, q such that $p - q = 2$. This is still unknown, but the past few years have seen a flurry of activity on this problem, beginning with Zhang's proof in 2013 that there are infinitely many primes whose gaps are bounded by some absolute constant.

In 2014 an alternative proof was given by Maynard, developing sieve theoretic techniques introduced by Goldston, Pintz, and Yildirim. Maynard's result states that there are infinitely many primes at most 600 apart.

The purpose of this essay is to explain Maynard's proof, and prove in detail the sieve theoretic aspects of this result. The Bombieri-Vinogradov theorem plays an important role, and so some discussion and proof of this should also be included. The focus should be on a high-level overview of the sieve techniques used by Goldston, Pintz, Yildirim, and Maynard.

Relevant Courses

Essential: Analytic Number Theory

References

[1] J. Maynard, "Small gaps between primes", Ann. of Math. (2) 181 (2015), 383–413.

[2] D. Goldston, J. Pintz, and C. Yildirim, "Primes in tuples. I." Ann. of Math. (2) 170 (2009), 819–862.

118. Arithmetic properties of random polynomials .... Dr P. P. Varjú

How likely is it that a random polynomial is irreducible? What is its Galois group? There are various settings where these questions may be asked. For example, one may consider the family of polynomials of fixed degree with coefficients belonging to growing subsets of the integers. Alternatively, one may restrict the coefficients to a fixed set and let the degree increase to infinity. References [2,4,5,7] discuss the first setting, while [1,3,6] consider the second one.

A successful essay will provide an overview of these results and discuss some of them in more detail with the necessary mathematical background. The essay writers has considerable freedom in choosing the focus of the essay, and they are recommended to consult the setter. Two possible choices are to focus on either of the settings mentioned above.

Relevant Courses

No courses are required but basic knowledge of Number Theory is very useful for this essay.

References

[1] Bary-Soroker, L. ; Kozma, G. *Irreducible polynomials of bounded height.* https://arxiv.org/abs/1710.05165

[2] Bhargava, M. ; Shankar, A. ; Wang X. *Squarefree values of polynomial discriminants I.* https://arxiv.org/abs/1611.09806

[3] Breuillard, E. ; Varjú P. P. *Irreducibility of random polynomials of large degree.* https://arxiv.org/abs/1810.13360

[4] Chela, R. *Reducible polynomials.* J. London Math. Soc. 38 (1963) 183–188.

[5] Chow, S. ; Dietmann, R. *Enumerative Galois theory for quartics.* https://arxiv.org/abs/1807.05820

[6] Konyagin, S. V. *On the number of irreducible polynomials with* $0, 1$ *coefficients.* Acta Arith. 88 (1999), no. 4, 333–350.

[7] Rivin, I. *Galois Groups of Generic Polynomials.* https://arxiv.org/abs/1511.06446

119. Computational methods for the linear Schrödinger equation .... Professor A. Iserles

This essay is concerned with the numerical solution of the linear Schrödinger equation in the semiclassical regime,

$$\varepsilon \mathrm{i}\frac{\partial u}{\partial t} = \varepsilon^2 \Delta u - V(x,t)u.$$

The main difficulty originates in the rapid oscillation generated by the small parameter $\varepsilon > 0$ and by the need to conserve energy.

In the last few years a number of methods have been proposed which exploit Lie-algebraic formalism to solve this equation. These methods combine operatorial splittings, Magnus-type expansions, symmetric Baker–Campbell–Hausdorff formula and spectral methods in a manner that leads to a very precise, yet affordable solution which preserves unitarity and other desirable features of the underlying equation.

The purpose of this essay is to review these developments, focussing on a single space dimension and periodic boundary conditions and highlighting problems with time-dependent interaction potential V.

Relevant Courses

Essential: Numerical solution of differential equations

References

[1] P. Bader, A. Iserles, K. Kropielnicka & P. Singh, "Effective approximation for the linear time-dependent Schrödinger equation", *Found. Comp. Maths* **14** (2014), 689–720.

[2] Bader, A. Iserles, K. Kropielnicka & P. Singh, "Efficient methods for time-dependence in semiclassical Schrödinger equations", *Proc. Royal Soc. A* (2016) DOI: 10.1098/rspa.2015.0733.

[3] Blanes, S. & Casas, F., *A Concise Introduction to Geometric Numerical Integration.* Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2016.

[4] A. Iserles, H.Z. Munthe-Kaas, S.P. Nørsett & A. Zanna, "Lie-group methods", *Acta Numerica* **9** (2000), 215–365.

120. Variational Autoencoders .... Dr S. A. Bacallado

Variational Autoencoders (VAE) [1][2] are a popular approach to perform variational inference using deep neural networks, producing complex generative models and powerful latent representations of the data. Applications include de-noising, image inpainting, music interpolation, semi-supervised learning and reinforcement learning.

This essay would provide an overview of VAE, as introduced in [1][2][3]. Then it could illustrate their application to a specific domain, perhaps by implementing a model on a chosen dataset. It could also discuss some improvements and extensions, such as those in [4][5][6].

Relevant Courses

Essential: Bayesian Modelling and Computation

References

[1] Diederik Kingma and Max Welling. *Auto-encoding variational bayes.* arXiv:1312.6114, 2013.

[2] Danilo Jimenez Rezende, Shakir Mohamed, and Daan Wierstra. *Stochastic backpropagation and approximate inference in deep generative models.* arXiv:1401.4082, 2014.

[3] Carl Doersch. *Tutorial on variational autoencoders.* arXiv:1606.05908, 2016.

[4] Anders Boesen Lindbo Larsen, Søren Kaae Sønderby, Hugo Larochelle, and Ole Winther. *Autoencoding beyond pixels using a learned similarity metric.* arXiv:1512.09300, 2015.

[5] Irina Higgins, Loic Matthey, Xavier Glorot, Arka Pal, Benigno Uria, Charles Blundell, Shakir Mohamed, and Alexander Lerchner. *Early visual concept learning with unsupervised deep learning.* arXiv:1606.05579, 2016.

[6] Francesco Paolo Casale, Adrian Dalca, Luca Saglietti, Jennifer Listgarten, and Nicolo Fusi. *Gaussian process prior variational autoencoders.* In Advances in Neural Information Processing Systems, pages 10390–10401, 2018.

121. R_0 ....

Professor J. R. Gog

The 'basic reproduction ratio', or R_0, is one of the key ideas in infectious disease dynamics: the mean number of secondary cases from an initial case in an otherwise susceptible population. This number depends on the disease and the population, and is often described as essential in understanding how to control an infectious disease. In the most basic mathematical models of disease, R_0 is easy to derive from basic parameters. However, for models with any extra complexities, such as a heterogeneous host population, it turns out that identifying R_0 can be less than obvious.

This essay should consider definitions and methods of calculating R_0 from a range of disease models. The student should then choose some direction(s) for further exploration. Here are some suggestions: how useful is R_0 in describing disease dynamics; the challenges of estimating R_0 for a real outbreak; ambiguities in the definitions of R_0; how R_0 might depend on population size and structure.

Reference [2] gives an overview to the field, and [1] gives a starting point for consideration of the reproductive ratio.

Relevant Courses

Essential: None

Useful: There are ways of drawing in your knowledge from a range of courses.

References

[1] Heffernan, J.M., Smith, R.J. and Wahl, L. M. (2005) Perspectives on the basic reproductive ratio *J. R. Soc. Interface* **2** 281–293 doi:10.1098/rsif.2005.0042

[2] Keeling, M.J. and Rohani, P., Modeling infectious diseases in humans and animals (Princeton University Press, 2007)

122. The Second Law of Quantum Complexity and the Entanglement Wormhole .... Dr J. Bausch

Quantum complexity arises as an alternative measure to the Fubini metric of the distance between two quantum states. Given two states and a set of allowed gates, it is defined as the least complex unitary operator capable of transforming one state into the other. Starting with K qubits evolving through a k-local Hamiltonian, it is possible to draw an analogy between the quantum system and an auxiliary classical system with 2k degrees of freedom. Using the definition of complexity to write a metric for the classical system, it is possible to relate its entropy with the quantum complexity of the K qubits, hence defining a Second law of Quantum Complexity. [1][2] The law states that, if it is not already saturated, the quantum complexity of a system will increase with overwhelming probability towards its maximum value. In the context of AdS/CFT duality and the ER=EPR conjecture, the volume of a certain maximal spacelike slice, which extends into the Black Hole interior, is proportional to the computational complexity of the instantaneous state of the conformal field theory. [3][4] Therefore, the interior of the Wormhole/Black Hole connecting two entangled CFT will grow as a natural consequence of the complexification of the boundary state.

Relevant Courses

Essential: Quantum Information Theory, Black Holes, String Theory

Useful: AdS/CFT, Advanced Quantum Field Theory

References

[1] A. R. Brown, L. Susskind, The Second Law of Quantum Complexity, arxiv:1701.01107

[2] A. R. Brown, L. Susskind, Ying Zhao, Quantum Complexity and Negative curvature, arxiv:1608.02612

[3] A. R. Brown, D. A. Roberts, L. Susskind, B. Swingle, Y. Zhao, Complexity equals Action, arxiv:1509.07876

[4] L. Susskind, Computational Complexity and Black Hole horizons, arxiv:1402.5674

[5] M. Van Raamsdonk, Building up spacetime with quantum entanglement, arXiv:1005.3035

[6] M. Van Raamsdonk, Lectures on Gravity and Entanglement, arXiv:1609.00026v1

[7] D.Harlow, Jerusalem Lectures on Black Holes and Quantum Information, arXiv:1409.1231v4

123. Combinatorial Morse Theory .... Dr H. Wilton

Classical Morse theory (as developed in [4]) is a way of understanding the topology of a manifold M via a *Morse function* $M \to \mathbb{R}$. Combinatorial Morse theory is an analogue of this theory for cell complexes, developed in parallel by Forman and Bestvina–Brady.

The latter were able to use it to provide counterexamples to various longstanding problems in topology: they constructed an infinitely presented group of type FP_2, and showed that at most one of the Whitehead and Eilenberg–Ganea conjectures hold [1].

The goal of this essay is to describe Bestvina–Brady's construction, and to give a proof of their main theorem, along with appropriate background material [2, 3] .

Relevant Courses

Essential: Part II Algebraic topology

Useful: Topics in Geometric Group Theory

References

[1] Mladen Bestvina and Noel Brady. Morse theory and finiteness properties of groups. *Inventiones Mathematicae*, 129:445–470, 1997.

[2] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[3] Kai-Uwe Bux and Carlos Gonzalez. The Bestvina–Brady construction revisited: geometric computation of σ-invariants for right-angled Artin groups. *Journal of the London Mathematical Society. Second Series*, 60:793–801, 1999.

[4] J. Milnor. *Morse theory.* Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51. Princeton University Press, Princeton, N.J., 1963.

124. Quantum random walks ....
Professor G. R. Grimmett

The classical random walk (RW) is very well understood in probability theory. The quantum random walk (QRW) is a distinct but related type of object with different short-time and long-time behaviour from that of RW. A successful essay will include clear accounts of the following topics: (i) different models for QRW, and their inter-relationships, (ii) walks in discrete and continuous time, (iii) limiting laws for QRW.

The emphasis throughout should be upon the communication of basic ideas and their relevance and context, rather than on giving full technical details.

References

[1] A. Ambainis, E. Bach, A. Nayak, A. Vishwanath, and J. Watrous, One-dimensional quantum walks, Proceeding STOC '01 Proceedings of the thirty-third annual ACM symposium on Theory of Computing, pages 37–49.

[2] N. Konno, Quantum random walks in one dimension (2002), arxiv.org/abs/quant-ph/0206053.

[3] G. Grimmett, S. Janson, and P. F. Scudo, Weak limits for quantum random walks, Physical Review E 69 (2004) Paper 026119.

[4] M. Szegedy, Quantum speed-up of Markov chain based algorithms (2004), FOCS '04 Proceedings of the 45th Annual IEEE Symposium on Foundations of Computer Science, pages 32–41.

[5] E. Farhi and S. Gutmann, Quantum computation and decision trees (1998), arxiv.org/abs/quant-ph/9706062

[6] N. Konno, H. J. Yoo. Limit theorems for open quantum random walks. J. Stat. Phys. 150 (2013), 299-319.

125. Endomorphisms of abelian varieties ....
Professor J. Thorne

Abelian varieties are the higher-dimensional analogues of elliptic curves in algebraic geometry, being projective group varieties of arbitrary dimension. They have many properties in common with elliptic curves, especially from an arithmetic point of view: for example, an abelian variety over $\mathbb{Q}$ has a finitely generated group of rational points (the Mordell–Weil theorem).

The aim of this essay will be to develop the basic theory of abelian varieties over a general field, including a discussion of projective embeddings and of the dual abelian variety, and to prove one of the fundamental theorems in the arithmetic of abelian varieties, namely Tate's theorem on endomorphisms of abelian varieties over finite fields.

References

[1] Mumford, Abelian Varieties. Tata Institute of Fundamental Research Studies in Mathematics.

[2] Tate, Endomorphisms of abelian varieties over finite fields. Invent. Math.

[3] Mumford. On the equations defining abelian varieties. I. Invent. Math.

126. Anyons and Topological Quantum Computation ....
Dr B. Béri

Noise and decoherence are key hurdles for prospective implementations of quantum computation. Among the suggested approaches to overcome these, topological quantum computation offers a paradigm that is intrinsically immune to such disturbances, up to corrections that are exponentially small in the system size.

Topological quantum computation is based on certain exotic quasiparticles in two spatial dimensions called anyons. Braiding anyon worldlines in three-dimensional spacetime amounts to performing certain logic gates on the state of the system; the concrete operations thus implemented depend only on the anyon type and topological class of the anyon trajectories. These gates are, therefore, independent of the local details of the process, which is a key feature behind the robustness of this quantum computational model.

The purpose of this essay is to describe anyons and their use for topological quantum computation. This should start with an account of the algebraic theory of anyons, including illustrations of the concepts on concrete abelian and nonabelian anyon models. On this basis, the essay should discuss topological quantum computation with Ising anyons, including approaches to make them computationally universal.

Relevant Courses

Essential: None

Useful: Part II Quantum Information and Computation, Part III Quantum Computation

References

[1] A. Kitaev, *"Anyons in an exactly solved model and beyond"*, Ann. Phys. **321**, 2 (2006).

[2] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, *"Non-Abelian Anyons and Topological Quantum Computation"*, Rev. Mod. Phys. **80**, 1083 (2008).

[3] B. A. Bernevig and T. Neupert, *"Topological Superconductors and Category Theory"*, arXiv:1506.05805.

[4] M. Barkeshli, C.-M. Jian, and X.-L. Qi, *"Genons, twist defects, and projective non-Abelian braiding statistics"*, Phys. Rev. B **87**, 045130 (2013).

[5] T. Karzig, Y. Oreg, G. Refael, and M. H. Freedman, *"A geometric protocol for a robust Majorana magic gate"*, Phys. Rev. X **6**, 031019 (2016).

127. g-expectations and risk measures ....

Dr M. Tehranchi

Peng [4] introduced a type of nonlinear expectations called g-expectations based on Backward Stochastic Differential Equations (BSDEs) and depending on a functional g. g-expectations have been applied to many areas in finance including pricing for contingent claims.
A variety of risk measures have been proposed in literature to evaluate future losses and to give indications about the acceptability of financial positions. In [2], Rosazza Gianin attempted to extend the risk measures into a nonlinear setting via g-expectations.
In this essay we investigate the connections between g-expectations and risk measures by focusing on two static risk measures. The starting point of the essay would be to give an introduction to BSDEs and g-expectations maybe including representation theorems and comparison theorems. The main part of the essay is supposed to focus on coherent and convex risk measures and properties for g-expectations in order to learn how g-expectations provide the families of risk measures mentioned above.

Relevant Courses

Essential: Stochastic Calculus and Applications

Useful: Advanced Probability, Advanced Financial Models

References

[1] Briand, P., Coquet, F., Hu, Y., Mémin, J., Peng, S. A converse comparison theorem for BSDEs and related properties of g-expectation. *Electronic Communications in Probability*, 5: 101-117, 2000.

[2] Rosazza Gianin, E. Risk measures via g-expectations. In *Insurance: Mathematics and Economics*, 39(1): 19-34, 2006.

[3] Jiang, L. Convexity, translation invariance and subadditivity for g-expectations and related risk measures. *The Annals of Applied Probability*, 18(1): 245-258, 2008.

[4] Peng, S. Backward SDE and related g-expectations. In *Backward Stochastic Differential Equations*, El Karoui, N., Mazliak, L. (Eds.). In *Pitman Research Notes in Mathematics Series*, 364: 141-159, 1997.

[5] Peng, S. Backward Stochastic Differential Equation, Nonlinear Expectation and Their Applications. In *Proceedings of the International Congress of Mathematicians 2010 (ICM 2010)*, 1: 393432, 2010.

128. Category $\mathcal{O}$ and Soergel bimodules ....

S. Martin

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$. An important subcategory of well-behaved infinite-dimensional $\mathfrak{g}$-modules is the so-called *category* $\mathcal{O}$. Unlike in the finite-dimensional case, modules in category $\mathcal{O}$ are generally not semisimple; in fact, category $\mathcal{O}$ is often considered a prototype for such behaviour. As such efforts towards understanding category $\mathcal{O}$ have been a guiding force in representation theory for several decades. One of the most famous results in this area is the Kazhdan–Lusztig conjecture, which describes the characters of simple modules in category $\mathcal{O}$. The original proof of this result around 1980 (due to Beilinson–Bernstein and Brylinski–Kashiwara independently) is a triumph of geometric representation theory, giving a precise correspondence between category $\mathcal{O}$ and perverse sheaves on the flag variety.

In the 1990s Soergel gave another version of this correspondence using an algebraic "bridge" between representation theory and geometry. More precisely, he constructed a category of bimodules over a polynomial ring which is closely related to both category $\mathcal{O}$ and a category of perverse sheaves on the flag variety. Soergel used these bimodules (today called Soergel bimodules) in a simplified, more algebraic proof of the Kazhdan–Lusztig conjecture. In addition to providing algebraic proofs of representation-theoretic statements, Soergel bimodules also generalize more easily to situations where the relevant geometric objects are poorly behaved or non-existent. For this reason Soergel bimodules are a very active area of research today.

The goal of this essay is to prove Soergel's correspondence between category $\mathcal{O}$ and Soergel bimodules. This can be done either in the original form, where projective objects in $\mathcal{O}$ correspond to Soergel modules, or in the more modern form, where wall-crossing functors in $\mathcal{O}$ correspond to Soergel bimodules. It will be necessary to learn enough of the general theory of category $\mathcal{O}$ to understand this correspondence, e.g. from [1]. Besides Soergel's first paper [2] on the topic, another good source is [4], which covers the theory of Soergel bimodules over arbitrary Coxeter systems. For the closely analogous modular case see [3].

Relevant Courses

Essential: Lie Algebras and their Representations

Useful: Algebra, Modular Representation Theory

References

[1] J. E. Humphreys. *Representations of semisimple Lie algebras in the BGG category $\mathcal{O}$*, volume 94. American Mathematical Soc., 2008.

[2] W. Soergel. Kategorie $\mathcal{O}$, perverse garben und moduln über den koinvarianten zur weylgruppe. *Journal of the American Mathematical Society*, 3(2):421–445, 1990.

[3] W. Soergel. On the relation between intersection cohomology and representation theory in positive characteristic. *Journal of Pure and Applied Algebra*, 152(1-3):311–335, 2000.

[4] W. Soergel. Kazhdan–lusztig-polynome und unzerlegbare bimoduln über polynomringen. *Journal of the Institute of Mathematics of Jussieu*, 6(3):501–525, 2007.

129. Hochschild cohomology of group algebras and crossed products .... Dr C.J.B. Brookes

Hochschild conomology is the appropriate cohomology theory for associative algebras and is closely related to defomration theory.

The initial aim of this essay would be to describe the work of Gerstenhaber [1] on the various algebraic structures on the Hochschild cohomology of an associative algebra before considering the particular case of nodular group algebras of finite groups. The abelian case was first considered by Holm [2] and Cibils and Solotar [3]. the general case was considered by Siegel and Witherspoon [3]. If space allows it would be good to consider more general examples of crossed products - see [5].

Relevant Courses

Essential: Part III Algebra

Useful: Modular representation theory

References

[1] M. Gerstenhaber, 'The cohomology structure of an associative ring', Ann. of Math. 78 (1963) 267288.

[2] T. Holm, 'The Hochschild cohomology ring of a modular group algebra: the commutative case', Comm. Algebra 24 (1996) 1957-1969.

[3] C. Cibils and A. Solotar, 'Hochschild cohomology algebra of abelian groups', Arch. Math. 68 (1997) 1721.

[4] S. F. Siegel and S. J. Witherspoon, The Hochschild cohomology ring of a group algebra, Proc. London Math. Soc. (3) 79 (1999), 131157.

[5] S.J. Witherspoon, Products in Hochschild cohomology and Grothendieck rings of group crossed products, Adv. Math. 185 (2004), no. 1, 136-158.

130. Gröbner bases and rewriting systems for associative algebras .... Dr C.J.B. Brookes

This topic is concerned with the existence of certain good algorithms in computational algebra, relating some questions from theoretical computer science to algebraic topology.

Gröbner bases were first defined by Buchberger in 1965 when he obtained an algorthm giving particularly well chosen generating sets for ideals in (commutative) polynomial algebras, and they have become a key tool in computational commutative algebra. The algorithms involved are linked to the construction of good free resolutions. The theory was first extended to consider ideals in free (non-commutative) algebras r by Bergman [3] and his work was extended by Mora. The existence of a Groebner basis for an ideal I of R corresponds to the existence of a good rewriting system for the quotient R/I.

I suggest the essay should start with a quick overview of Buchberger's work for commutative polynomial algebras before considering some of the non-commutative developments. The first two references discuss the commutative case. The paper of Kobayashi discusses the construction of free resolutions in the non-commutative case given a Gröbner basis and would be one option for the non-commutative part of the essay.

Relevant Courses

Essential: Part III Algebra

Useful:

References

[1] W.W. Adams, P. Loustaunau, An introduction to Gröbner bases, Graduate studies in mathematics 3, American Mathematical Society 1994.

[2] T. Becker, V. Weispfennig, Gröbner bases, a computational approach to commutative algebra, Graduate texts in mathematics 141, Springer 1993.

[3] G. M. Bergman, The diamond lemma for ring theory, Adv. in Math., 29 (1978), 178218.

[4] Y. Kobayashi, Gröbner bases of associative algebras and the Hochschild cohomology, Transactions of the AMS, 357, 1095 - 1134.

131. Machine Learning for Classification of Astronomical Time Series ...... Dr K. S. Mandel

The night sky is replete with astronomical sources that change in brightness over time, including variable stars, gravitational lensing events, and stellar explosions, such as supernovae and kilonovae. The Large Synoptic Survey Telescope (LSST) is a new 8-meter telescope being constructed in Chile that will achieve first light in 2019 and commence its 10-year main survey in 2022. It will regularly scan the sky and record brightness time series (light curves) of millions of time-varying sources in multiple colours of light, revolutionising our understanding of these astrophysical phenomena. Astronomers and data scientists have engineered a variety of machine learning algorithms to automatically sift through the massive data streams and classify the variable and transient sources underlying the time series data. The Photometric LSST Astronomical Time Series Classification Challenge (PLAsTiCC) is an open data challenge to the community to develop and apply new methods to classify simulated astronomical time-series data in preparation for observations from LSST. This data challenge poses the question: how well can we classify objects in the sky that vary in brightness from realistic simulated LSST time-series data, with all its observational challenges and non-representativity?

This essay is meant to review the relevant scientific motivations and challenges, the statistical issues involved, some of the algorithms that have been developed, their pros and cons, and the metrics by which their performance are evaluated. The student will have the opportunity to implement the method(s) of his or her choice or invention on the datasets available from the PLAsTiCC challenge posted on Kaggle (https://www.kaggle.com/c/PLAsTiCC-2018) and evaluate their performance, either for the general challenge, or for more focused scientific goals. Originality and creativity are encouraged.

Relevant Courses

Essential: Astrostatistics

References

[1] The PLAsTiCC Team, et al. *The Photometric LSST Astronomical Time-series Classification Challenge (PLAsTiCC): Data set.* 2018, LSST DESC Note. https://arxiv.org/abs/1810.00001.

[2] Malz, A., et al. *The Photometric LSST Astronomical Time-Series Classification Challenge (PLAsTiCC): Selection of a Performance Metric for Classifiation Probabilities Balancing Diverse Science Goals.* 2018, https://arxiv.org/abs/1809.11145.

[3] Narayan, G., et al. *Machine-learning-based Brokers for Real-time Classification of the LSST Alert Stream.* 2018, The Astrophysical Journal Supplement, 236 9.

[4] Lochner, M. et al. *Photometric Supernova Classification with Machine Learning.* 2016, The Astrophysical Journal Supplement, 225, 31.

[5] Richards, J. et al. *Semi-supervised learning for photometric supernova classification.* 2012, Monthly Notices of the Royal Astronomical Society, 419, 1121.

[6] Kessler, R. et al. *Results from the Supernova Photometric Classification Challenge.* 2010, Publications of the Astronomical Society of the Pacific, 122, 1415.

132. At least 1/3 of the Riemann zeros lie on the critical line .... Dr T. Bloom

Description: The location of the zeros of the Riemann zeta function is one of the biggest mysteries of number theory, and a deep understanding of their distribution has led to many important results. The infamous Riemann Hypothesis states that all of the non-trivial zeros s such that $\zeta(s) = 0$ lie on the critical line $\Re(s) = 1/2$. Hardy proved the weaker result that there are infinitely many zeros on the critical line.

Selberg improved this to show that a positive proportion of all non-trivial zeros lie on the critical line, and then Levinson obtained the more precise conclusion that at least 1/3 of the zeros lie on the critical line (so the Riemann hypothesis is, in some sense, at least 1/3 true!) This aim of this essay is to understand the arguments of Hardy, Selberg, and Levinson, and give an exposition explaining the main ideas and how improvements were obtained. If time permits, it could also briefly discuss the more recent results of Conrey and Bui-Conrey-Young, improving the constant in Levinson's result.

Relevant Courses

Essential: Analytic Number Theory

References

[1] H. Bui, B. Conrey, and M. Young, "More than 41% of the zeros of the zeta function are on the critical line", Acta Arith. 150 (2011), 35–64.

[2] N. Levinson, "More than one third of zeros of Riemann's zeta-function are on $\sigma = 1/2$", Advances in Math. 13 (1974), 383–436.

[3] A. Selberg, "On the zeros of Riemann's zeta-function", Skr. Norske. Vid. Akad. Oslo I. (1942)

[4] E. Titchmarsh, "The Theory of the Riemann Zeta-Function", Clarendon Press (1951)

133. Study of some rigorous results on the vortices approximation of the 2-D Euler equations . . . . Professor C. Mouhot

This essay will study old and new results about the vortices approximation of 2-D incompressible fluid dynamics. This problem has its root in the point-vortex model of Helmholtz, Kirchhoff, studied by Onsager in the context of 2-D hydrodynamic turbulence. The mathematical challenge consists in proving rigrously the convergence of the vortex model to the 2-D Euler or Navier-Stokes equations in the many-particle limit (mean-field regime). This is connected to the functional space where stability holds for the limiting nonlinear PDE (the vorticity formulation of the 2D incompressible Euler equation), to entropy estimates and to large deviation estimates.

The essay will present a state of the art of the mathematical results on this question, detailed proofs of the main results in the recent papers and preprints, and outline the common concepts.

Relevant Courses

Essential: Analysis, Linear Analysis, Introduction to PDEs

Useful: Statistics, Fluid mechanics

References

[1] J. Goodman, T. Hou, J. Lowengrub, *Convergence of the point vortex method for the 2-D Euler equations*, Comm. Pure Appl. Math. 43 (1990), no. 3, 415–430.

[2] J.-M. Delort, *Existence de nappes de tourbillon en dimension deux. (French) [Existence of vortex sheets in dimension two]* J. Amer. Math. Soc. 4 (1991), no. 3, 553–586.

[2] S. Schochet, *The weak vorticity formulation of the 2-D Euler equations and concentration-cancellation*, Comm. Partial Differential Equations 20 (1995), no. 5-6, 1077–1104.

[3] S. Schochet, *The point-vortex method for periodic weak solutions of the 2-D Euler equations*, Comm. Pure Appl. Math. 49 (1996), no. 9, 911–965.

[3] S. Serfaty, *Mean Field Limit for Coulomb Flows*, Preprint available at: https://math.nyu.edu/~serfaty/submittedversion.pdf.

[4] P.-E. Jabin, Z. Wang, *Quantitative estimates of propagation of chaos for stochastic systems with $W^{1,\infty}$ kernels*, Invent. Math. 214 (2018), no. 1, 523–591.